Chain Drift in Leading Superlative Indexes

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by

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In a recent paper, Hill [2006] compares the performance of a large class of superlative indexes. He finds that most of these indexes yield strikingly different results as the linking interval used to obtain chained estimates of growth over a specified period lengthens. Samuelson and Swamy [1974] had shown that, with nonhomothetic preferences, some amount of chain drift–that is, difference between the rate of change calculated by chaining an index over a multi-period interval and that obtained using endpoints only–is inevitable. Nevertheless, Hill showed that three superlative indexes, the Fisher, the Tornqvist, and the Walsh, are less susceptible to chain drift as the linking interval lengthens than the others he considered. The Fisher and Tornqvist are by far the most commonly used superlative indexes. The Fisher index is used, in particular, to calculate the real aggregates shown in the national income and product accounts of the United States (NIPA's). This paper extends Hill's results by showing that, under realistic assumptions, the Fisher index is more susceptible to chain drift than is the Tornqvist. Empirical examples confirm this possibility.

Using data for the period 1967-2002, we illustrate the alternative measures of long-term real growth that result from different index number formulas and different linking intervals. The calculations are for real GDP and real gross private domestic investment (GPDI). Using the Fisher and the implicit Tornqvist index number formulas, almost identical results are obtained when the linking interval is a quarter or a year, the linking intervals used in the NIPA's. However, as the linking interval lengthens, the Fisher index estimates diverge substantially while the divergence among the implicit Tornqvist estimates is much smaller.

This finding motivates the paper's theoretical contributions. The signs of the chain drift for Fisher price and quantity indexes are determined from simple statistical assumptions, and the signs obtained (for the Fisher quantity index, lower growth rates for chained than for binary estimates) agree with the empirical finding. An analysis of the Fisher and Tornqvist formulas provides insight into the greater stability of the Tornqvist estimates. This analysis confirms and extends Lent's [2000] finding of only small chain drift in the Tornqvist price index.

If chain indexes were always calculated with short linking intervals, then the results of this paper would be of little importance. When linking intervals are short, the Fisher and Tornqvist give very similar results. However, if an index is constructed from, say, decennial Census data, the results of this paper suggest a preference for the Tornqvist. Moreover, it is not clear that a short linking interval is always best, even when high frequency data are available. In the index number literature, chain drift is identified as a problem when prices or quantities oscillate rather than trend (Forsyth and Fowler [1981] and Szulc [1983]). Bouncing behavior in prices or business cycles in quantities exacerbate the differences in the estimates from different linking intervals. Forsyth and Fowler, and Szulc, suggest a variable linking interval, with longer intervals used when oscillatory data are identified. A formal procedure for calculating optimal linking intervals is proposed in Ehemann [2005] and, because the optimal linking interval is occasionally found to be long, the Tornqvist index is used in illustrative calculations for national aggregates.

The format of this paper is as follows. In Section 1, we present a numerical example of chain drift and examine its extent empirically using data for the U. S. economy. In Section 2, we show that the differences between the chain and binary Fisher price and quantity indexes result from the correlation between changes in price and the changes in expenditure shares attributable to changes in income. Section 3 shows that such correlation need not imply chain drift in the Tornqvist index and provides further comparison with the Fisher. Section 4 concludes. An Appendix contains the proof of a theorem in Section 2 on the chain drift of the Fisher index.

1. Some chained and binary growth rates

An index number formula is transitive if chaining from t = 1 to t = 2 and then from t = 2 to t = 3 gives the same index value for the index at t = 3 as the direct index from t = l to t = 3. Thus transitivity is equivalent to the absence of chain drift. Fixed-weighted indexes are transitive, but they have been discarded in many applications because they have been shown to exhibit serious substitution bias. The testing of an index number formula for transitivity by determining whether the chained and direct calculation of the index value are equal is known as a circularity test. The chained Fisher and Tornqvist indexes satisfy this test in two very special cases: (1) if the path taken by the data can be divided into two parts, with the second an exact retrace of the first, and (2) if the data are generated by a consumer with homothetic preferences, so that expenditure shares remain constant as income changes with relative prices fixed.

Table 1 provides a numerical example of the failure, in general, of Fisher and Tornqvist indexes to satisfy transitivity. There are two goods and four periods. Prices of the two goods and the Fisher chain quantity index are set equal to 1.00 in year zero. In each subsequent period, substitution effects are the normal ones; that is, the quantity of each good changes in the opposite direction from the change in its relative price. In year three, prices and quantities are the same as in year zero. The circularity test requires that the index return to its original value of 1.00. However, the computed value of the Fisher index is 34 percent higher. For the Tornqvist index, the value in year three is 13 percent higher than in year zero. The test again fails by a wide margin, but by less than for the Fisher index. The binary Fisher and Tornqvist indexes based to year zero, of course, take the "reasonable" value of 1.00 in year three.

These results are more extreme than those observed in practice. In the remainder of this section we compare chained and binary estimates of growth rates in real GDP and real gross private domestic investment using two index number formulas, the Fisher quantity index and the implicit Tornqvist quantity index. The results for the Fisher and implicit Tornqvist indexes using data for the U. S. economy are qualitatively consistent with those of the numerical example. There are substantial differences between the Fisher chain and binary estimates and smaller differences between the implicit Tornqvist chain and binary estimates. The differences between chain and binary are especially large for the Fisher estimates of real gross private domestic investment.

The Fisher chain and binary indexes are defined as follows. Let P_j be a vector of period *j* prices and Q_j the corresponding vector of period *j* quantities. Then the binary Fisher quantity index for period *t* compared with period *0* is

(1) $$Q_t^{FB} = \left(\frac{P_0 Q_t}{P_0 Q_0} \frac{P_t Q_t}{P_t Q_0} \right)^{1/2} .$$

The chain index is built up as the product of quantity relatives for the consecutive periods *(j-l, j)* that are subintervals of *(0, t)*, each having the same functional form as (1), and is thus defined to be

(2) $$Q_t^{FC} = \prod_{j=1}^{j=t} \left(\frac{P_{j-1} Q_j}{P_{j-1} Q_{j-1}} \frac{P_j Q_j}{P_j Q_{j-1}} \right)^{1/2} .$$

The effect of chaining with the Fisher quantity index on real GDP growth rates is shown in Table 2, columns 2 and 3. Column 2 shows the average rate of growth in real GDP for the thirty year period 1967-1997, estimated using alternative intervals for chaining. The linking intervals are quarterly and one, three, and ten years. The binary estimate over thirty years is also shown. Column 3 shows the level of the GDP quantity index in 1967 for the same alternatives, assuming the value of the index in the reference year 1997 is set to 100. The starting date 1967 was selected because a major change in the source data for exports and imports occurred in that year. The level of detail is held fixed at that available quarterly in 1967 in order to gauge the effect of chaining alone. The published estimates of real GDP growth, shown for comparison, are based on annual chaining and use the maximum level of underlying detail available for each year-to-year interval. There is little difference between the estimated average growth rates obtained by chaining quarterly and at one and three year intervals, but the estimates of the average growth rate over the period change as the linking interval lengthens further. The estimated average annual growth rate is 2.8 percentage points higher when the linking interval is thirty years than when linking is quarterly.

Table 2, columns 4 and 5, presents the corresponding evidence on the effect of chaining for one of the major components of real GDP, real gross private domestic investment. This component is selected because of its economic importance and because the spectacular decline in computer prices over the period makes the measurement of this component particularly sensitive to the methods employed. For this component, changing the length of the linking interval is seen to have a noticeable effect on the growth rate even when these intervals are relatively short, while the estimated average growth rate more than doubles, from 4.02 percent to 8.32 percent, as the linking interval goes from quarterly to thirty years. The test approach to index number theory provides no guidance as to which of these average growth rates is “correct.”

The binary form of the Tornqvist quantity index is

$$Q_t^{TB} = \prod_{i=0}^{i=n} \left(\frac{q_{it}}{q_{i0}} \right)^{(s_{i0}+s_{it})/2} ,$$

where the q_{ij}, (for $i = 1, 2, ..., n$ and $j = 0, t$) are the components of Q_j, the s_{ij} are the corresponding expenditures shares, $p_{ij}q_{ij}/\Sigma_h p_{hj}q_{hj}$, and the p_{ij} are the components of P_j. The chained form of the Tornqvist quantity index is

$$Q_t^{TC} = \prod_{i=0}^{i=n} \prod_{j=1}^{j=t} \left(\frac{q_{ij}}{q_{i,j-1}} \right)^{(s_{i,j-1}+s_{ij})/2} .$$

Although the Tornqvist index is widely used, it is not available as a potential measure of real GDP. This index is undefined for the interval (0, t) when the quantity of a component good is zero in period 0. This problem occurs frequently in the underlying data for real GDP, in particular in the detailed data for Federal defense purchases. The Tornqvist index is also undefined when the quantity of a component good differs in sign in the two periods, as frequently occurs in the data for change in private inventories.

The implicit Tornqvist quantity index is a closely related index for which these problems do not occur. An implicit quantity index is obtained by dividing the ratio of nominal expenditures in two specified periods by a price index for the same two periods; the implicit Tornqvist quantity index is obtained by dividing the ratio of nominal expenditures by the Tornqvist price index. Thus, the binary implicit Tornqvist quantity index is given by

$$Q_t^{IB} = \frac{P_t Q_t}{P_0 Q_0} \prod_{i=0}^{i=n} \left(\frac{p_{it}}{p_{i0}} \right)^{-(s_{0i}+s_{ti})/2}$$

and the chain implicit Tornqvist quantity index is given by

$$Q_t^{IC} = \prod_{j=1}^{j=t} \frac{P_j Q_j}{P_{j-1} Q_{j-1}} \prod_{i=0}^{i=n} \left(\frac{p_{it}}{p_{i0}} \right)^{-(s_{i,j-1}+s_{ij})/2} .$$

The Tornqvist price index avoids the problems encountered in attempting to measure real GDP using the Tornqvist quantity index because inventory prices are positive and reasonable procedures are available to impute a positive price to a good when no transactions occur in a particular period. (Such imputations are also required by the Fisher quantity index.). The implicit Tornqvist thus emerges as a possible alternative to the Fisher index for multi-period estimates of real GDP.

The performance of direct and implicit quantity indexes has been compared by Allen and Diewert [1981]. They recommend a superlative direct quantity index when there is less variation in quantity ratios than price ratios. However, for the case of less variation in price ratios, they recommend an implicit quantity index based on a superlative price index.

Table 3 repeats the analysis of Table 2 using the implicit Tornqvist index instead of the Fisher. The amount of chain drift over the thirty year period is reduced substantially. The difference between the largest and smallest estimated growth rates is less than 0.2 percentage point for real GDP and less than 0.6 percentage point for real gross private domestic investment. Moreover, while the Fisher estimated average growth rates always increase as the linking interval is lengthened, the implicit Tornqvist estimated average growth rates show no such pattern.

2. *Chain drift and the Fisher index*

Forsyth and Fowler [1981] investigated chain drift in the Laspeyres, Paasche, geometric, and Fisher indexes. They found that, just as the value of the (binary) Fisher index always lies between those of the Laspeyres and Paasche indexes, the chain drift of the Fisher index is always between that of the Laspeyres and Paasche. The chain drifts for the latter indexes were opposite in sign and thus at least partially offsetting. Szulc [1983] showed, further, that the signs of chain drift in the Laspeyres and Paasche price indexes depend on the pattern of serial correlation in the prices. If price changes from the preceding period are positively correlated with the relevant cumulative price changes and consumers respond normally to changes in relative prices, the chained Laspeyres price index tends to be smaller than the corresponding binary Laspeyres index and the chained Paasche price index tends to be larger than the corresponding binary Paasche index. Lent [2000] investigated chain drift in the geometric and Tornqvist price indexes. She found that the sources of chain drift in the Tornqvist index tended to be offsetting, so that chain drift could be expected to be small with no presumption as to sign. For the geometric price index, on the other hand, if prices and expenditure shares have consistent trends and the elasticity of price substitution is between zero and one, the chained index will be greater than its binary counterpart. This section presents a corresponding result for chain drift in the Fisher index that helps to explain the behavior of this index as shown in the preceding section.

First, however, we provide some perspective on chaining and chain drift. A clearly desirable property for price aggregates is that the quantities or expenditures used to weight the price changes of individual goods be contemporaneous with them. Because consumers respond to price changes by changing quantities purchased (usually in the direction opposite of the price change), the use of other than contemporaneous quantities or expenditures introduces a bias, substitution bias. A substitution bias also arises in the measurement of quantity aggregates when the quantity changes of individuals goods are combined using weights that are not contemporaneous. Contemporaneous measurement of the prices, quantities, and/or expenditures appearing in an index number formula requires chaining. In fact, the principle implies that substitution bias is minimized when the linking interval is as short as possible. A second reason for chaining is that it permits the inclusion of new goods and the dropping of discontinued goods as data for these goods become available or disappear. If this were the whole story, chain drift would not present a problem. True, different results would be obtained from different linking intervals (or from a binary estimate over some longer period), but one could confidently choose the estimate that used the shortest linking interval. The difficulty is that high frequency economic data may introduce other sources of error into the chaining process. Measurement error due to the smaller sample sizes underlying much high frequency data and short-term oscillatory behavior in prices or quantities can degrade the quality of chained estimates.

The ubiquity of chain drift stems from nonhomothetic preferences. Samuelson and Swamy [1974] demonstrated that with nonhomothetic preferences, chain indexes cannot be transitive. Nevertheless, although contradicted by empirical evidence, the assumption of homothetic preferences plays a key role in the economic theory of index numbers, as in the derivation of the Fisher and Tornqvist indexes from the optimizing behavior of consumers having homothetic utility functions (as in Diewert [1976]).[2] Forsyth and Fowler [1981] suggested a practical response to this dilemma: if intransitivity cannot be eliminated from empirical estimates, it might nevertheless be reduced by choosing one index number formula rather than another. As observed above, previous authors have identified the Fisher and Tornqvist indexes as ones for which chain drift could be expected to be smaller than for many other index number formulas. The investigation of the Fisher index in this section lays groundwork for comparison of the two index number formulas on this criterion.

We investigate chain drift for the Fisher price index under conditions tailored to the question of measuring long term growth. To obtain definite results, we make six assumptions. First, we assume that the rate of change in price for each good is constant over time. This is the limiting case of Szulc's [1983] assumption that price changes over time are positively correlated. Second, we assume that there exists at least one pattern of income over time for which the rate of change in expenditure for each good is constant and we choose one of these patterns of income as a baseline. Third, realized income exceeds baseline income. Fourth, we assume that the main variables in the analysis – rates of change in prices, rates of change in expenditures for individual goods, and the initial expenditure shares – are distributed lognormally across goods.[3] We do not

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[2] A difficulty in dispensing with the homotheticity assumption is that it can be violated in many different ways, leading to many different index number formulas, each exact for some nonhomothetic utility function. Nevertheless, Feenstra and Reinsdorf [2000] provide an exact price index for an important nonhomothetic case, the almost ideal demand system (AIDS).

[3] Strictly, expenditure shares cannot have the lognormal distribution because they cannot exceed one. However, the lognormal distribution has a thin right-hand tail. If the number of goods is large, the expenditure shares will be concentrated around a very small mean and the thin right-hand tail will provide an acceptable approximation. Use of a truncated lognormal distribution, for example, would add unwarranted complexity.

require that the parameters of the joint lognormal distribution remain constant from one period to the next. Fifth, we require a large number of goods.

Finally, we make an important assumption about the sign of a parameter in the assumed lognormal distribution: that there is a positive correlation between a good's rate of change in price and the rate of change in expenditure on that good resulting from an increase in income. This assumption can be rationalized in terms of more basic assumptions about price and income changes and the utility function of a representative consumer. For the utility function, it is assumed that (1) the elasticities of substitution for all pairs of goods lie between zero and one, and (2) along all possible expansion paths (as income increases) the expenditure share for any particular good either increases continuously or decreases continuously. (The elasticities of substitution used here are not standard ones: they are income compensated to maintain rates of change in expenditures, not to maintain utility levels.) In each period, income increases beyond the baseline level according to some probability law, with small increases having nonzero probability. Now, as a result of the price changes in a given period, the consumer moves to a new expansion path. Consider a good whose relative price has increased. By the elasticity of substitution assumption, the expenditure share of this good will increase with no change in income beyond the baseline change. If the new expansion path is one for which the expenditure share for this good is increasing (for a randomly chosen good, a 50 percent probability), the expenditure share increases for any increase in income. If the new expansion path is one for which the expenditure share is decreasing, then by the elasticity of substitution assumption, the expenditure share will increase if the further increase in income is sufficiently small, which has positive probability. Thus, on balance, the correlation between changes in expenditure shares and changes in prices is positive. The correlation between expenditure changes and changes in prices will also be positive because, when relative changes are considered, the change in expenditure shares equals the change in expenditures times a ratio of total expenditures that is the same for all goods.

Theorem. Suppose that the prices of all goods grow at constant rates, not all equal. Let

$k_i = (p_{ij}q_{ij})/(p_{i,j-1}q_{i,j-1})$ be the ratio of expenditure on good i in period j to that in the preceding period, assumed constant over periods $j = 1, 2, ..., t,$ given a baseline income path Y_j, $j = 0, 1, 2, ..., t,$

v_{ij} = the change in k_i that results from an increase in income from its baseline path; that is, $v_{ij} = \partial k_i / \partial Y_j^* = \partial k_i / \partial \varepsilon_j$, where $Y_j^* = Y_j + \varepsilon_j$, $\varepsilon_j > 0$,

w_i = the expenditure share of good i in period 0, and

$x_i = p_{ij}/p_{i,j-1}$ be the constant ratio of the price of good i to that in the preceding period.

Assume that the v_{ij}, w_i, and x_i, are jointly lognormally distributed and that the v_{ij} and x_i are positively correlated. Then, if the number of goods is sufficiently large, the Fisher chain price index will increase more rapidly (or decrease less rapidly) than the Fisher binary price index over the period $j = 0$ to $j = t$.

The theorem is proved in the Appendix. In the homothetic case, if all prices increase at the same rate (so that relative prices are constant), expenditures on all goods will change at the same rate, depending on the rate of change in income. The theorem states that when this special case is excluded, an increase in income results in a greater change in the Fisher chain price index than in the Fisher binary price index, provided that the other conditions stated in the theorem are satisfied.

Corollary 1. Suppose that quantities and prices grow at constant rates, not all equal, and that the other conditions of the theorem also hold. Then the chained Fisher chain quantity index will increase less rapidly (or decrease more rapidly) than the Fisher binary quantity index over the same time interval.

Corollary 1 follows from the fact that the Fisher quantity index can be obtained by dividing expenditures, indexed to the reference year, by the Fisher price index. This result is consistent with the rising calculated growth rates, as the linking interval lengthens, shown for real GDP and real GPDI in Table 2.

Alternatively, suppose that the realized income path Y_j^* grows at rates smaller than required to maintain the constant rates of expenditure growth k_i, $i = 1, ..., n$. That is, $Y_j^* = Y_j + \varepsilon_j$, $\varepsilon_j < 0$. Then the v_{ij} are negative. We have

Corollary 2. Adopt the same assumptions as the theorem, except that (i) $\varepsilon_j < 0$, (ii) the $-v_{ij}$, w_i, and x_i, are jointly lognormally distributed, and (iii) the $-v_{ij}$ and x_i are positively correlated. Then the Fisher chain price index will increase less rapidly (or decrease more rapidly) than the Fisher binary price index over the period $j = 0$ to $j = t$. The Fisher chain quantity index will increase more rapidly (or decrease less rapidly) than the binary Fisher quantity index over the same time interval.

Corollary 2 is proved by redefining u_{ij} in Appendix equation (A.9) as $u_{ij} = -v_{ij}w_i$.

To reverse the results of the theorem, we have assumed that income growth is less than that required to maintain constant rates of growth in expenditures, which appears just as plausible as the theorem assumption that it is greater. It may be that in strong rates of growth in the relevant income measures underlie the empirical results reported in Table 2. However, Corollary 2 further assumes a negative correlation between price change and income compensated expenditure change, which is usually contradicted empirically. The chain drift shown for real GDP and real private domestic investment in Table 2 as the linking interval lengthens could also be the net result of positive chain drift when income shocks are positive and mixed outcomes when they are negative.

3. Fisher vs. Tornqvist

The results of the preceding section comparing Fisher chain and binary indexes can be compared with corresponding properties of Tornqvist chain and binary indexes. Lent [2000] has shown, using algebra similar to that used to derive the corresponding ratio of Fisher indexes in Appendix equation (A.7), that the ratio of the chain Tornqvist price index to the binary Tornqvist price index over the same interval is

$$(3) \qquad \frac{P^{TC}}{P^{TB}} = \left(\prod_{i=1}^{n} \prod_{j=1}^{t-1} \left[\left(\frac{p_{it}}{p_{ij}} \right)^{s_{ij}-s_{i,j-1}} \left(\frac{p_{i0}}{p_{ij}} \right)^{s_{i,j+1}-s_{i,j}} \right] \right)^{1/2} .$$

Lent states that if prices and shares are steadily increasing or decreasing, the "drift factors" in (3) should neutralize each other. This conjecture can be established formally.

As in the case of the Fisher indexes, we assume constant rates of price change $x_i = p_{ij}/p_{i,j-1}$. Thus, $p_{ij} = p_{i0}x_i^j$ for all i and j. However, for the Fisher index, it was analytically convenient to represent departures from hometheticity by assigning goods unequal constant-over-time growth rates in expenditures. For the Tornqvist index, departures from hometheticity are most easily represented by arithmetically constant increases or decreases in shares. Hence we assume $s_{ij} = s_{i,j-1} + c_i$ for all i and j, where $\Sigma c_i = 0$. Squaring both sides of (3) and making these substitutions, we obtain

$$(4) \qquad \left(\frac{P^{TC}}{P^{TB}} \right)^2 = \prod_{i=1}^{n} \prod_{j=1}^{t-1} \left[x_i^{(t-j)c_i} x_i^{-jc_i} \right]$$

$$= \prod_{i=1}^{n} \prod_{j=1}^{t-1} \left[x_i^{(t-2j)c_i} \right].$$

Taking logarithms of both sides, we obtain

$$(5) \qquad 2(\ln P^{TC} - \ln P^{TB}) = \sum_{i=1}^{n} \sum_{j=1}^{t-1} (t-2j) c_i \ln x_i$$

$$= \sum_{i=1}^{n} c_i \ln x_i \sum_{j=1}^{t-1} (t-2j)$$

$$= 0$$

because $\sum_{j=1}^{t-1} j = \frac{t}{2}$. Thus, in this simple but significant nonhomothetic case, the Tornqvist chained and binary indexes give identical results.

This demonstration for the Tornqvist index and the theorem for the Fisher index in the preceding section illustrate differences in the propensity for chain drift of the two indexes but do not indentify the underlying sources of these differences. This task occupies the remainder of this section. Reinsdorf [1996] and Reinsdorf, Diewert, and Ehemann [2002] showed that the Fisher index could be expressed as a geometric index with time-varying expenditure weights. Because the Tornqvist is also a geometric index with time-varying expenditure weights, this form facilitates comparison of the two indexes.

Following Reinsdorf, the geometric index form of the binary Fisher price index is

$$P^{FB} = \prod_{i=1}^{n} x_i^{.5(\gamma_i + \delta_i)}$$

where

$$\gamma_i = \frac{s_{i0}\, m(x_i, \bar{x})}{\sum_{h=1}^{n} s_{h0}\, m(x_h, \bar{x})} \qquad \text{and} \qquad \delta_i = \frac{s_{i0}\, m(y_i, \bar{y})}{\sum_{h=1}^{n} s_{h0}\, m(y_h, \bar{y})},$$

$m(a,b)$ denotes the logarithmic mean:

$$m(a,b) = \frac{a - b}{\ln a - \ln b},$$

and

$$y_i = 1/x_i, \qquad \bar{x} = \Sigma s_{i0} x_i, \quad \text{and} \quad \bar{y} = \Sigma s_{it} y_i.$$

Note that $\bar{x}$ is the Laspeyres price index and $\bar{y}$ is the reciprocal of the Paasche price index.

The logarithm of the geometric form of the Fisher index can be approximated by that of the Tornqvist index plus additional terms. To show this, we replace γ_i by a linear Taylor's series approximation in terms of $m(x_i, \bar{x})$ and evaluated at $m(x_h, \bar{x}) = \bar{x}$, $h = 1, 2, ..., n,$ and we replace δ_i by a linear Taylor's series approximation in terms of $m(y_i, \bar{y})$ and evaluated at $m(y_h, \bar{y}) = \bar{y}$, $h = 1, 2, ..., n.$ Because the numerators and denominators of the logarithmic means go to zero at these evaluation points, the latter must be approached as limits. Define the vectors $X = (x_1, x_2, ..., x_n)$, $\bar{X} = (\bar{x}, \bar{x}, ..., \bar{x})$, $Y = (y_1, y_2, ..., y_n)$, $\bar{Y} = (\bar{y}, \bar{y}, ..., \bar{y})$, $M_X = (m(x_i, \bar{x}), ..., m(x_n, \bar{x}))$, and $M_Y = (m(y_i, \bar{y}), ..., m(y_n, \bar{y}))$. Then, because $lim_{a \to b}\, m(a,b) = b$, $X \to \bar{X}$ and $Y \to \bar{Y}$ imply $M_X \to \bar{X}$ and $M_Y \to \bar{Y}$, the limits required by the Taylor's series expansions.

Consider first only the zero-order Taylor's series terms in the expansions of γ_i and $\delta_{i.}$ We have

$$\lim_{X \to \bar{X}} \gamma_i = s_{i0} \quad \text{and} \quad \lim_{Y \to \bar{Y}} \delta_i = s_{it}$$

for $i = 1, 2, ..., n.$ Thus, to order zero, the Fisher index is approximated by the Tornqvist:

$$P^{FB} \approx \prod_{i=1}^{n} x_i^{.5(s_{i0}+s_{it})}.$$

Turning to the first-order terms, we have

$$\frac{\partial \gamma_i}{\partial m(x_i,\bar{x})} = \gamma_i s_{i0} \left(\frac{1}{m(x_i,\bar{x})} - \frac{s_{i0}}{\Sigma_h s_{h0} m(x_i,\bar{x})} \right)$$

and

$$\frac{\partial \delta_i}{\partial m(y_i,\bar{y})} = \delta_i s_{i0} \left(\frac{1}{m(y_i,\bar{y})} - \frac{s_{i0}}{\Sigma_h s_{h0} m(y_i,\bar{y})} \right),$$

so that

$$\lim_{X \to \bar{X}} \frac{\partial \gamma_i}{\partial m(x_i,\bar{x})} = s_{i0}^2 \frac{1 - s_{i0}}{\bar{x}}$$

and

$$\lim_{Y \to \bar{Y}} \frac{\partial \gamma_i}{\partial m(y_i,\bar{y})} = s_{it}^2 \frac{1 - s_{it}}{\bar{y}}.$$

Combining these results, the logarithm of the binary Fisher price index is approximated by

(6)
$$\ln P^{FB} \approx \frac{1}{2} \sum_{i=1}^{n} (\ln x_i) \left[s_{i0} + s_{it} + s_{i0}^2 (1 - s_{i0}) \left(\frac{m(x_i,\bar{x})}{\bar{x}} - 1 \right) + s_{it}^2 (1 - s_{it}) \left(\frac{m(y_i,\bar{y})}{\bar{y}} - 1 \right) \right].$$

Thus, the Fisher price index differs from the Tornqvist according to the extent that goods with large expenditure shares are also goods whose price change differs from the average. The two indexes will differ, in particular, if the rate of price change for an important good is an "outlier." However, a good with an above average x_i will have a below average y_i. Consequently, the first-order terms are partially offsetting. For a single good to have a large effect on the net difference between the two indexes, there must also be a large change in the good's expenditure share between the two periods.

An approximation of the Fisher quantity index as a Tornqvist quantity index plus additional terms can be derived in exactly the same way as (6). If $f_i = q_{it}/q_{i,t-1}$ is assumed constant over time and we define $g_i = 1/f_i$, $\bar{f} = \Sigma s_{i0} f_i$, and $\bar{g} = \Sigma s_{i0} g_i$,

$$\text{(7)} \qquad \ln Q^{FB} \approx \frac{1}{2}\sum_{i=1}^{n} (\ln f_i)\left[s_{i0} + s_{it} + s_{i0}^2(1 - s_{i0})\left(\frac{m(f_i, \bar{f})}{\bar{f}} - 1 \right) + s_{it}^2(1 - s_{it})\left(\frac{m(g_i, \bar{g})}{\bar{g}} - 1 \right) \right].$$

The Fisher quantity index differs from the Tornqvist according to the extent that goods with large expenditure shares are also goods whose quantity change differs from the average. Referring to equation (7), one can consider the differences between the Fisher and Tornqvist quantity indexes reported in Tables 2 and 3. When the linking interval was small, the Fisher and Tornqvist gave almost identical results. This corresponds to a finding that the third and fourth terms in the large brackets in (7) are nearly offsetting. Suppose now that the linking interval is *k* periods. In place of f_i we have f_i^k and, similarly, the replacements g_i^k for g_i, $\bar{f}^{[k]} = \Sigma s_{i0} f_i^k$ for $\bar{f}$, and $\bar{g}^{[k]} = \Sigma s_{i0} g_i^k$ for $\bar{g}$. The quantities $\bar{f}^{[k]}$ and $\bar{g}^{[k]}$ are in general quite different from their $\bar{f}$ and $\bar{g}$ counterparts. The first is much more affected by the goods whose rates of quantity change is greatest, while the second is much more affected by the goods whose rates of quantity change is smallest. The result is that the tendency for the third and fourth terms in (7) to offset is much weaker and the Fisher and Tornqvist differ.

This finding illustrates a more general result reported by Hill [2006]. The Tornqvist and Fisher indexes are members of the quadratic-mean-of-order-r class of superlative indexes, with $r = 0$ giving the Tornqvist and $r = 2$ the Fisher. Hill found that the weight given to outliers by members of the class increases with *r*.

Equation (7) also provides a perspective on the differences between Fisher estimates based on different linking intervals. Consider the ratio of chain to binary Fisher price indexes over the same interval. If we return to the very special case in which expenditure shares change arthimetically as prices change geometrically, the ratio P^{FC}/P^{FB} will approximately equal the ratio P^{TC}/P^{TB}, which by equations (4) and (5) is equal to unity, times an expression involving f_i, f_i^k, g_i, g_i^k, and their corresponding means. Whenever important components have divergent rates of growth, the terms in the long-interval chain will fail to cancel. The geometric form of the Fisher index is not convenient for sorting out the net effect, but that was done, given some additional assumptions, in the theorem of the preceding section.

4. Summary and conclusions

It is generally believed that the Fisher and Tornqvist indexes will give very similar results. This belief is the result of experience with short linking intervals. We have shown that as the linking interval of chained Fisher and Tornqvist indexes is lengthened, the two index types diverge. In tests using data for real GDP and real private domestic investment, the Fisher index diverged from estimates using quarterly or annual linking more rapidly than did the Tornqvist, and in further contrast to the Tornqvist, all the changes were in the same direction.

It is also generally believed that chain drift is a property of indexes that attempt to measure aggregates exhibiting oscillatory or cyclical behavior. An extreme numerical example of oscillatory behavior illustrated this source of chain drift. However, we showed analytically that the Fisher index of an aggregate displays chain drift even when prices and expenditures of the individual goods change at constant or nearly constant rates. We also showed that, under appropriate assumptions (including constant rates of price change), the direction of chain drift observed in the Fisher estimates of real GDP and real GPDI could be predicted.

Although all chained indexes will exhibit chain drift in the absence of homothetic preferences, we showed that the Tornqvist index does not always require homothetic preferences to be free of chain drift. By approximating the logarithm of the Fisher index as the logarithm of the Tornqvist index plus additional terms, we were able to identify some of the factors that increase the propensity to chain drift in the Fisher.

In view of these results, it remains true that if chaining is done with short linking intervals, the issue of chain drift does not affect the choice between the Fisher and the Tornqvist. That choice can be made on other grounds. However, if for some reason longer linking intervals must be used, these results suggest that the Tornqvist index should be strongly considered. Moreover, short linking intervals do not eliminate the possibility of chain drift, which can also appear as a result of bounces and cycles in the data. Ehemann [2005] investigates the extent of such chain drift in the NIPA's and, where found, adjusts for it using the chained Tornqvist with longer links.

Appendix:
Proof of the Theorem on Fisher Index Chain Drift

We first prove two lemmas that establish inequalities that follow from the bivariate lognormal distribution. Then we present a theorem that establishes the relative magnitudes of chain and binary Fisher price indexes.

Lemma 1. Let v, w, and x be jointly distributed lognormal random variables and let (v_i, w_i, x_i) for $i = 1, 2, ..., n$ be a sample from this distribution. Define the random variable $u = vw$ with sample values $u_i = v_i w_i$. Assume that v and x are positively correlated (i.e., $\rho(v, x) > 0$) and let t be a positive integer greater than one. Then, for sufficiently large values of n,

(A.1) $$\frac{\Sigma u_i x_i^{-(t+1)}}{\Sigma w_i x_i^{-(t+1)}} - \frac{\Sigma u_i x_i^{-t}}{\Sigma w_i x_i^{-t}} > \frac{\Sigma u_i x_i^{-t}}{\Sigma w_i x_i^{-t}} - \frac{\Sigma u_i x_i^{-(t-1)}}{\Sigma w_i x_i^{-(t-1)}}.$$

Because taking logarithms of variates does not change the sign of their correlation, $\rho(\ln v, \ln x) < 0$. Let $\mu(\cdot, \cdot)$ denote a covariance, so that $0 < \mu(-\ln v, \ln x)$, and add $\mu(\ln v, -t \ln x)$ to both sides of this inequality. Recalling that the covariance of a sum is the sum of the covariance, this gives $\mu(\ln v, -t \ln x) < \mu(\ln v, -(t-1)\ln x)$. Multiplying both sides by 0.5 and taking exponentials yields

$$\exp(.5\,\mu(\ln v, -t\ln x)) < \exp(.5\,\mu(\ln v, (t-1)\ln x)).$$

Multiplying both sides of this inequality by the expression $\exp(.5\mu(-\ln v, -\ln x)) - 1$, which is negative, reverses the direction of inequality and yields

$$\exp(.5\,\mu(\ln v, -t\ln x))[\exp(.5\,\mu(\ln v, -\ln x))-1]$$
$$> \exp(.5\,\mu(\ln v, -(t-1)\ln x))[\exp(.5\,\mu(\ln v, -\ln x))-1].$$

Performing the multiplications gives

$$\exp(.5\,\mu(\ln v, -(t+1)\ln x)) - \exp(.5\,\mu(\ln v, -t\ln x))$$
$$> \exp(.5\,\mu(\ln v, -t\ln x)) - \exp(.5\,\mu(\ln v, -(t-1)\ln x)).$$

Now make the substitution $\ln v = \ln u - \ln w$ and again use the rule that the covariance of a sum is the sum of the covariances to obtain

$$\frac{\exp(.5\,\mu(\ln u, -(t+1)\ln x))}{\exp(.5\,\mu(\ln w, -(t+1)\ln x))} - \frac{\exp(.5\,\mu(\ln u, -t\ln x))}{\exp(.5\,\mu(\ln w, -t\ln x))}$$
$$> \frac{\exp(.5\,\mu(\ln u, -t\ln x))}{\exp(.5\,\mu(\ln w, -t\ln x))} - \frac{\exp(.5\,\mu(\ln u, -(t-1)\ln x))}{\exp(.5\,\mu(\ln w, -(t-1)\ln x))}.$$

Next multiply each side by the ratio $(E[u]\,E[1/x])/(E[w]E[1/x])$, where $E[\cdot]$ denotes expected value. We obtain

$$\text{(A.2)}\quad \frac{E[u]E[1/x]}{E[w]E[1/x]}\,\frac{\exp(.5\,\mu(\ln u, -(t+1)\ln x))}{\exp(.5\,\mu(\ln w, -(t+1)\ln x))} - \frac{E[u]E[1/x]}{E[w]E[1/x]}\,\frac{\exp(.5\,\mu(\ln u, -t\ln x))}{\exp(.5\,\mu(\ln w, -t\ln x))}$$
$$> \frac{E[u]E[1/x]}{E[w]E[1/x]}\,\frac{\exp(.5\,\mu(\ln u, -t\ln x))}{\exp(.5\,\mu(\ln w, -t\ln x))} - \frac{E[u]E[1/x]}{E[w]E[1/x]}\,\frac{\exp(.5\,\mu(\ln u, -(t-1)\ln x))}{\exp(.5\,\mu(\ln w, -(t-1)\ln x))}.$$

For random variables υ and ω having the bivariate lognormal distribution, the joint moments around zero are given by

$$E[\upsilon^{\alpha}\omega^{\beta}] = E[\upsilon]E[\omega]\exp(.5\alpha\beta\mu(\ln\upsilon,\ln\omega)),$$

where α and β are positive integers [Simon, 2002, p. 23, our notation]. In (A.2), we have $\omega = 1/x$, $\alpha = 1$, $\beta = t$, and, in separate expressions, $\upsilon = u$ or $\upsilon = w$. Because u is the product of lognormal variates, it is itself lognormal. Also, because it is the reciprocal of a lognormal variate, $1/x$ is lognormal. Thus, (A.2) can be written

(A.3) $$\frac{E[ux^{-(t+1)}]}{E[wx^{-(t+1)}]} - \frac{E[ux^{-t}]}{E[wx^{-t}]} > \frac{E[ux^{-t}]}{E[wx^{-t}]} - \frac{E[ux^{-(t-1)}]}{E[wx^{-(t-1)}]}.$$

Because the sample moments from a lognormal distribution are consistent estimators of the corresponding population moments, (A.1) follows from (A.3) provided that n is sufficiently large.

<u>Lemma 2</u> . Under the same assumptions as Lemma 1,

(A.4) $$\frac{\Sigma u_i x_i^{-2}}{\Sigma w_i x_i^{-2}} - \frac{\Sigma u_i x_i^{-1}}{\Sigma w_i x_i^{-1}} > \frac{\Sigma u_i x_i^{-1}}{\Sigma w_i x_i^{-1}} - \frac{\Sigma u_i x_i}{\Sigma w_i x_i}$$

provided that n is sufficiently large.

An inequality of the form $R_1 - R_2 > R_3 - R_4$ holds if

$$R_2\left(\frac{R_1}{R_2} - 1\right) > R_3\left(1 - \frac{R_4}{R_3}\right).$$

The inequality (A.4) has this form with $R_2 = R_3$, so it can be established by proving an inequality of the form

(A.5) $$R_1/R_2 - 1 > 1 - R_4/R_3.$$

Consider another inequality with the same form as (A.4),

(A.6) $$\frac{E[ux^{-2}]}{E[wx^{-2}]} - \frac{E[ux^{-1}]}{E[wx^{-1}]} > \frac{E[ux^{-1}]}{E[wx^{-1}]} - \frac{E[ux]}{E[wx]}.$$

Using the properties of moments around zero of the bivariate lognormal distribution, we have

$$\begin{aligned}\frac{R_1}{R_2} &= \frac{E[ux^{-2}]}{E[wx^{-2}]}\left(\frac{E[ux^{-1}]}{E[wx^{-1}]}\right)^{-1} \\ &= \exp(.5\,\mu(\ln v, -2\ln x) - .5\mu(\ln v, -\ln x)) \\ &= \exp(-.5\,\mu(\ln v, \ln x))\end{aligned}$$

and

$$\frac{R_4}{R_3} = \frac{E[ux]}{E[wx]}\left(\frac{E[ux^{-1}]}{E[wx^{-1}]}\right)^{-1}$$

$$= \exp(.5\mu(\ln v, \ln x) - .5\mu(\ln v, -\ln x))$$

$$= \exp(\mu(\ln v, \ln x)).$$

Taking series expansions of the exponential terms, we obtain

$$\frac{R_1}{R_2} - 1 = -.5\mu(\ln v, \ln x) + .125[\mu(\ln v, \ln x)]^2 - \ldots$$

and

$$1 - \frac{R_4}{R_3} = -\mu(\ln v, \ln x) - .5[\mu(\ln v, \ln x)]^2 - \ldots .$$

Thus, (A.5) holds for these values of R_1/R_2 and R_4/R_3 when $\rho(v, x)$ and therefore $\mu(\ln v, \ln x)$ are positive. Consequently, (A.6) holds. The inequality (A.4), the sample counterpart of (A.6), holds when n is sufficiently large.

The proof of the theorem consists primarily of recasting the ratio of the Fisher chain to the Fisher binary price index into a form to which lemmas 1 and 2 can be applied.

The Fisher binary price index, P^{FB}, and Fisher chain price index, P^{FC}, are defined by interchanging *p's* and *q's* in text equations (1) and (2), respectively. The squared ratio of these indexes is

$$\left(\frac{P^{FC}}{P^{FB}}\right)^2 = \frac{\prod_{j=1}^{t} \frac{Q_{j-1}P_j}{Q_{j-1}P_{j-1}} \frac{Q_jP_j}{Q_jP_{j-1}}}{\frac{Q_0P_t}{Q_0P_0} \frac{Q_tP_t}{Q_tP_0}}.$$

Some algebraic manipulation gives[4]

[4] The algebraic steps is this paragraph parallel the simplification of the ratio of the chained to the binary Tornqvist price indexes in Lent [2000].

$$\left(\frac{P^{FC}}{P^{FB}}\right)^2 = \prod_{j=1}^{t} \frac{\dfrac{Q_{j-1}P_j}{Q_{j-1}P_{j-1}} \dfrac{Q_t P_{j-1}}{Q_t P_j}}{\dfrac{Q_0 P_j}{Q_0 P_{j-1}} \dfrac{Q_j P_j}{Q_j P_{j-1}}}$$

$$= \prod_{j=1}^{t} \left(\frac{\sum \dfrac{p_j}{p_{j-1}} \dfrac{p_{j-1}q_{j-1}}{\Sigma p_{j-1}q_{j-1}}}{\sum \dfrac{p_j}{p_{j-1}} \dfrac{p_{j-1}q_0}{\Sigma p_{j-1}q_0}} \frac{\sum \dfrac{p_{j-1}}{p_j} \dfrac{p_j q_t}{\Sigma p_j q_t}}{\sum \dfrac{p_{j-1}}{p_j} \dfrac{p_j q_j}{\Sigma p_j q_j}} \right),$$

where the summations run over goods. The goods subscript i will be suppressed when clear from the context. Now let $x_i = p_{ij}/p_{i,j-1}$ denote the constant-over-time relative change in price of the ith good in consecutive periods and let $e_{imm^*} = p_{im}q_{im^*}$ denote "generalized expenditure" on the ith good, where m and m^* are particular periods, not necessarily the same. With this notation, the squared ratio of price indexes can be written

$$\left(\frac{P^{FC}}{P^{FB}}\right)^2 = \prod_{j=1}^{t} \left(\frac{\sum x \dfrac{e_{j-1,j-1}}{\Sigma e_{j-1,j-1}}}{\sum x \dfrac{e_{j-1,0}}{\Sigma e_{j-1,0}}} \frac{\sum x^{-1} \dfrac{e_{j,t}}{\Sigma e_{j,t}}}{\sum x^{-1} \dfrac{e_{j,j}}{\Sigma e_{j,j}}} \right).$$

In the first and last periods, this expression contains sums that are equal and thus cancel. Hence $(P^{FC}/P^{FB})^2$ can be written more compactly by advancing subscripts in the first ratio within the large brackets by one period and reducing the terms in the product by one. The result is

$$\left(\frac{P^{FC}}{P^{FB}}\right)^2 = \prod_{j=1}^{t-1} \left(\frac{\sum x \dfrac{e_{jj}}{\Sigma e_{jj}}}{\sum x \dfrac{e_{j0}}{\Sigma e_{j0}}} \frac{\sum x^{-1} \dfrac{e_{jt}}{\Sigma e_{jt}}}{\sum x^{-1} \dfrac{e_{jj}}{\Sigma e_{jj}}} \right).$$

In this expression, the denominators Σe_{jj} cancel, so we can write

$$\text{(A.7)} \qquad \left(\frac{P^{FC}}{P^{FB}}\right)^2 = \prod_{j=1}^{t-1} \left(\frac{\Sigma x^{-1} e_{jt}}{\Sigma e_{jt}} \frac{\Sigma x e_{jj}}{\Sigma x^{-1} e_{jj}} \frac{\Sigma e_{j0}}{\Sigma x e_{j0}} \right).$$

Let $k_i = e_{ijj}/e_{i,j-1,j-1}$ denote the constant-over-time ratio of expenditures on good i in consecutive periods. Let the expenditure share of good i in period zero, w_i, be given by

$$w_i = e_{i00}/\Sigma_h e_{h00} = e_{i00}/Y(0).$$

Then

$$e_{imn} = e_{i00} x_i^{m-n} k_i^n \;\; = w_i x_i^{m-n} k_i^n Y(0).$$

In this notation,

$$\left(\frac{P^{FC}}{P^{FB}}\right)^2 = \prod_{j=1}^{t-1} A_j B_j C_j,$$

where

$$A_j = \frac{\Sigma x^{-1} e_{jt}}{\Sigma e_{jt}} = \frac{\Sigma w x^{j-t-1} k^t}{\Sigma w x^{j-t} k^t}, \quad B_j = \frac{\Sigma x e_{jj}}{\Sigma x^{-1} e_{jj}} = \frac{\Sigma w x k^j}{\Sigma w x^{-1} k^j},$$

$$and \quad C_j = \frac{\Sigma e_{j0}}{\Sigma x e_{j0}} = \frac{\Sigma w x^j}{\Sigma w x^{j+1}}.$$

The logarithm of $(P^{FC}/P^{FB})^2$ can be approximated by a linear Taylor's series expansion around a point at which all the k_i's are equal. The first-order terms require the partial derivatives of $ln\,A_j$, $ln\,B_j$, and $ln\,C_j$ for $j = 1, 2, ..., t$ with respect to k_i for $i = 1, 2, ..., n$. These partial derivatives are, respectively,

$$\frac{\partial \ln A_j}{\partial k_i} = \sum_i \left(\frac{t w_i x_i^{j-t-1} k_i^{t-1}}{\Sigma w x^{j-t-1} k^t} - \frac{t w_i x_i^{j-t} k_i^{t-1}}{\Sigma w x^{j-t} k^t} \right),$$

$$\frac{\partial \ln B_j}{\partial k_i} = \sum_i \left(\frac{j w_i x_i k_i^{j-1}}{\Sigma w x k^j} - \frac{j w_i x_i^{-1} k_i^{j-1}}{\Sigma w x^{-1} k^j} \right),$$

and

$$\frac{\partial \ln C_j}{\partial k_i} = 0.$$

Evaluating these partial derivatives and the zero-order term of the Taylor's series expansion at $k_i = \bar{k}$, $i = 1, 2, ..., n$, we obtain the Taylor approximation

$$\ln\left(\frac{P^{FC}}{P^{FB}}\right)^2 \simeq \ln\left(\prod_{j=1}^{j=t-1} \frac{\Sigma wx^{j-t-1}}{\Sigma wx^{j-t}} \frac{\Sigma wx}{\Sigma wx^{-1}} \frac{\Sigma wx^{j}}{\Sigma wx^{j+1}}\right)$$

$$+ \sum_{j=1}^{j=t-1} \sum_{i=1}^{i=n} \frac{1}{\bar{k}}\left(\frac{t w_i x_i^{j-t-1}}{\Sigma wx^{j-t-1}} - \frac{t w_i x_i^{j-t}}{\Sigma wx^{j-t}} + \frac{j w_i x_i}{\Sigma wx} - \frac{j w_i x_i^{-1}}{\Sigma wx^{-1}}\right)(k_i - \bar{k}).$$

Now consider the effects of an increase in income in period j from its baseline path. Because, by assumption, $\partial k_i / \partial \varepsilon_j = v_{ij}$, the rate of change of $ln\ (P^{FC}/P^{FB})^2$ with respect to a change in total expenditures in period j is

(A.8) $$\frac{\partial \ln(P^{FC}/P^{FB})^2}{\partial \varepsilon_j} = \sum_{i=1}^{i=n} \frac{1}{\bar{k}}\left(\frac{t w_i x_i^{j-t-1}}{\Sigma wx^{j-t-1}} - \frac{t w_i x_i^{j-t}}{\Sigma wx^{j-t}} + \frac{j w_i x_i}{\Sigma wx} - \frac{j w_i x_i^{-1}}{\Sigma wx^{-1}}\right) v_{ij}.$$

To complete the proof, we show that if the number of goods, n, is sufficiently large, (A.8) is positive for all values of j. Let $u_{ij} = v_{ij} w_i$. Because $t \geq j$, it is sufficient to show that

(A.9) $$\frac{\Sigma_i u_j x^{j-t-1}}{\Sigma_i wx^{j-t-1}} - \frac{\Sigma_i u_j x^{j-t}}{\Sigma_i wx^{j-t}} > \frac{\Sigma_i u_j x^{-1}}{\Sigma_i wx^{-1}} - \frac{\Sigma_i u_j x}{\Sigma_i wx},$$

for all j and t. For any t and with $j = t - 1$, (A.9) holds by Lemma 2. Inequality (A.9) can be verified for $j = t - 2$ by combining lemmas 1 and 2. Specifically, upon setting $t = 1$ in (A.1), the right-hand side of (A.1) equals the left-hand side of (A.4), implying (A.9). Repeated substitutions from (A.1) with successively higher values of t establish (A.9) for all values of j. The theorem is proved.

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Table 1.–Example of the Circularity Problem

	Year 0	Year 1	Year 2	Year 3
Price of good 1	1.00	2.00	1.50	1.00
Quantity of good 1	3.00	2.00	2.20	3.00
Price of good 2	1.00	0.30	0.50	1.00
Quantity of good 2	2.00	5.00	1.70	2.00
Fisher quantity relative	--	1.17	0.67	1.70
Fisher chain quantity index	1.00	1.17	0.79	1.34
Tornqvist quantity relative	--	1.04	0.83	1.30
Tornqvist chain quantity index	1.00	1.04	0.86	1.13

Table 2.–Average Annual Growth Rates (Fisher) for Real Gross Domestic Product and Real Gross Private Domestic Investment, 1967-1997

Linking Interval	(1) Average Annual Growth Rate, GDP (percent)	(2) GDP Quantity Index for 1967 (Year 1997 = 100)	(3) Average Annual Growth Rate, GPDI (percent)	(4) GPDI Quantity Index for 1967 (Year 1997 = 100)
1967-level underlying detail:				
Quarterly	3.02	40.92	4.02	30.63
1 year	3.08	40.26	4.16	29.46
3 years	3.10	40.08	4.24	28.75
10 years	3.34	37.27	5.00	23.11
30 years	5.82	18.21	8.32	9.08
Full underlying detail:				
1 year (official estimate)	3.10	40.04	4.16	29.41

Note: Quarterly data for 1967 are unpublished deflation-level detail. Where additional detail are available annually or in subsequent years, Fisher aggregates are substituted to hold the level of detail constant. Estimates except those that chain quarterly are based on annual data.

Table 3.–Average Annual Growth Rates (Implicit Tornqvist) for Real Gross Domestic Product and Real Gross Private Domestic Investment, 1967-1997

Linking Interval	(1) Average Annual Growth Rate, GDP (percent)	(2) GDP Quantity Index for 1967 (Year 1997 = 100)	(3) Average Annual Growth Rate, GPDI (percent)	(4) GPDI Quantity Index for 1967 (Year 1997 = 100)
1967-level underlying detail:				
Quarterly	3.08	40.26	4.03	30.54
1 year	3.06	40.52	4.12	29.82
3 years	2.99	41.27	3.71	33.52
10 years	2.89	42.56	3.56	35.04
30 years	3.03	40.87	3.77	32.94
Full underlying detail:				
1 year	3.08	40.29	4.13	29.69

Note: Quarterly data for 1967 are unpublished deflation-level detail. Where additional detail are available annually or in subsequent years, Fisher aggregates are substituted to hold the level of detail constant. Estimates except those that chain quarterly are based on annual data.