Implications of Geometric Cohort Depreciation for Service-Life Distributions

Brian K. Sliker*
Bureau of Economic Analysis

September 25, 2018

Abstract

This paper decomposes cohort-level Geometric depreciation into a weighted average of individual non-Geometric forms. The individual forms are general enough to represent actual individual depreciation patterns adequately, and the weights on individuals resemble Gamma service-life densities under most circumstances. Researchers with a knowledge of used assets' retirement patterns, but not prices, could use the Gamma parameters to estimate the features of (stipulated) Geometric cohort depreciation. One of the individual forms, weighted appropriately, could smoothly replace Geometric depreciation when the rate of return is negative. The analysis also offers support for faster depreciation rates for structures assets in the National Income and Product Accounts.

JEL Codes: C65 (Miscellaneous Mathematical Tools), D24 (Production, Capital, and Productivity), M41 (Accounting)
Keywords: depreciation, declining-balance rate, service lives, levels of aggregation, flexible functional forms, Gamma distribution

---

*The author thanks Barbara Fraumeni, Arnold Katz, Dylan Rassier, and Jon Samuels for useful comments and suggestions. Errors and views in this paper are the author's own, not necessarily those of the Bureau of Economic Analysis. Author e-mail: brian.sliker@bea.gov. Address: U.S. Bureau of Economic Analysis, Suitland Federal Center, 4600 Silver Hill Road, Suitland, MD 20746. Tel: (301) 278-9649. To replicate the numerical results of section 3, please contact the author for the *Excel* workbook "FindLDists.xlsx."

0. Introduction

This paper carries out a bottoms-up reassembly of the constant-rate, or geometric, model of depreciation that is commonplace in production economics. It situates geometric depreciation for any given asset type at a fairly high level of aggregation — i.e., the "cohort," which comprises all the individuals of the type that are produced or installed within a fairly narrow, well-defined time-frame, and that typically have some technical characteristics in common. (Think of an automobile model-year, or even a variety — say, Ford *Windstar LX*s — within a model-year.) Constant-rate depreciation at all cohort levels is sufficient for depreciation at that same rate of the entire *net stock* of the type, which combines all the cohorts of the asset-type. This is an important milestone, for it facilitates the treatment of the net stock as a unitary aggregate, without the complications of "vintage accounting" whereby each cohort is tracked separately.[1]

The individuals that make up a cohort may follow decidedly non-geometric depreciation patterns, and they are removed from service at different ages. Engineering studies of how individuals wear out are rare in economics, though the discounting procedures that would convert loss of usefulness to loss of resale value are well understood. This paper offers three families of individual-level functional forms for depreciation and their associated forms for serviceability. One family is especially compatible with geometric cohort-level depreciation, via the use of a two-parameter Gamma distribution of retirement ages to weight across individuals. Those same two parameters are closely linked to a short-hand technique that statistical agencies often use to assign a constant depreciation rate to an asset type when little more is known about the type than its average service life. A second family can be analytically construed as reasonably approximating the resale value and serviceability patterns of *any* asset-type at the individual level (so the missing engineering studies aren't critical). A third family allows capital accountants to "switch horses" from the common geometric model when the rate of return goes negative, which has been a problem for productivity measurement for the past decade.

The paper's main contribution, though, is intuitive. The three individual-level families' functional *shapes* strongly resemble each other over most of the relevant resale-price and service-flow spaces, so arguments that are precisely correct for one of the families are about right for the other two. In particular, the use of *approximately Gamma* numerical retirement distributions can reconcile individual profiles from the second and third families to geometric cohort-level depreciation, and the good approximating abilities of the second family imply that nearly-Gamma numerical retirement distributions can be constructed to reconcile almost any individual-level depreciation family to a geometric cohort. The contrapositive is the rub. If an asset type's retirements are not remotely Gamma distributed, then we may doubt whether its cohorts depreciate at a constant rate. So the Gamma retirement distribution encodes much of what economists ask about geometric depreciation.[2]

---

[1] A unitary aggregate with constant-rate depreciation allows the simplest form of the well-known Perpetual Inventory Model (PIM) of capital. See Diewert and Wei (2017) for another way to get constant depreciation at the net-stock level, depending strongly on a constant growth rate of real investment over long stretches of years. Whether the simplest PIM is so administratively desirable anymore in an era of ubiquitous computing is arguable.

[2] Eighty years on, Winfrey's (1935) survival distributions are still the best known. Some resemble Gamma forms.

A side-benefit of emphasizing the connection between individuals and their cohort is a better sense of "obsolescence" and how to measure it apart from ordinary wear-and-tear. For a narrow enough definition of a cohort, one individual cannot be obsolete relative to another, because both have the same technical characteristics. In fact, when the two individuals were brand new, they were essentially identical, and their sales-prices also must have been nearly equal, or at least *more* equal than later, after years of unequal use. On the other hand, a new member of last year's cohort, stored in an ideal vault away from dust, vermin, moisture, oxygen, and children, would still likely sell for less today than a new member of this year's cohort, fresh off the assembly line. The ratio of as-if-new to *actually* new prices at the same date would constitute a crisp measure of obsolescence within an overall asset-type. The ideal vault is not available, but one might approximate its effects by a careful hedonic age-price regression, so long as an individual's cohort membership was not collinear with its age and date of resale. This paper does not carry out such a regression, which is put off to an ongoing empirical project, but it does work out how to treat obsolescence and ordinary depreciation jointly, and it properly treats anticipated obsolescence as part of the effective, "own" rate of return.

The remainder of the paper has four major sections. The first uses a well-worn tool from accounting — i.e., to estimate an asset-type's constant depreciation rate, just divide its average service-life into a "declining balance rate" typically set to 1 or 2 but never really explained — to motivate a family of individual-level resale-price profiles that, properly weighted, integrate cleanly to a geometric profile at the cohort level. Both the individual profiles and their (Gamma-distributed) weights are fairly general, so the prospects for widespread cohort geometric depreciation look good. Moreover, the declining-balance parameter is revealed to have a deeper, two-sided meaning — as a gauge of how well an individual holds its value before it is retired, and as the "shape parameter" of the Gamma retirement distribution — and it may take any value at least as great as 2. (Later we'll work that down to 1).

The second section introduces two other individual-level resale-price families. Both correct a deficiency in the service-flows that one might predict from the math of the first family. Both are also general enough to approximate the resale-price and service-flow patterns of nearly any reasonable individual-level profile, where a few loose arguments for "reasonability" will be offered.

The third section constructs numerical distributions of service-lives to reconcile the two corrective resale-price families with cohort-level geometric depreciation. By varying the values of the system's known or assumed parameters — e.g., the cohort depreciation rate, the "curvature" parameter of the individual-level resale-price family, the rate of return, and the range and granularity of the admissible grid of ages and service-lives — one can back out the density that "makes things work." This section rehearses several such exercises, and all of them but two return densities very near to Gamma forms. The key exception is when the rate of return is negative. I will then argue that geometric cohort depreciation can fail in liquidity-trap conditions.

The fourth major section integrates two pacings of obsolescence — unexpected shock and persistent-rate (i.e., geometric) — into the three individual-level depreciation and deterioration families. Provided service-life densities are slow-to-move, sudden-shock obsolescence would require occasional resets of the cohort-level resale-price function, while persistent (hence expected) geometric obsolescence would both speed cohort depreciation and add to the effective discount rate.

Altogether, the paper is an applied-math effort to help practicing productivity economists and capital accountants better understand notions they had long taken for granted (and therefore wouldn't touch), so it builds and uses models and graphs, but there are no data here. Nonetheless, two short sections edge toward applicability. Section 5 examines the origins of BEA's depreciation estimates for 1-to-4 unit residential structures, which the analyses of Sections 2 and 3 suggest are too slow. Section 6 offers an exit ramp from geometric cohort capital accounting when rates of return are zero or negative. A conclusion recounts the arguments. Three appendixes work with Gamma distributions, derive the three resale-price and service-flow families from forms available in accounting textbooks, and show the near-Gamma service-life densities of asset types whose service-flows meet precipitous ends.

1. What's a Declining Balance Rate, Really?

Capital accountants often use the average service life $\sqsubset$ of an asset type, but *no actual used-price information*, to estimate the type's constant geometric annual depreciation rate, simply by dividing $\sqsubset$ into a fixed and seemingly arbitrary number known as the *Declining Balance Rate* (DBR):

$$d = \text{DBR} / \sqsubset \tag{1.1}$$

The resulting depreciation rate d ($0<d<1$) then informs the value-relative-to-new ratio of the collection of s-year-old assets of that type — i.e., the age-s "cohort" — as:[3]

$$\Theta_{\text{discrete}} = (1-d)^s \tag{1.2}$$

For an equivalent continuous-time calculation of the value-ratio, the transformation $d = 1-e^{-\delta}$ allows:

$$\Theta(s) = e^{-\delta s} \tag{1.3}$$

A continuous-time version v of the DBR is:

$$\delta = v / \sqsubset \tag{1.4}$$

so the algebra relating v and DBR follows as:

$$v = DBR \times \delta/d = DBR \times \delta/(1-e^{-\delta}) \approx \text{DBR} \times (1+\delta/2) \tag{1.5}$$

In the math that follows, I'll work with v and δ, but it is easy to convert back to DBR and d.

For most equipment asset types tracked by the U.S. Bureau of Economic Analysis (BEA), the appropriate declining balance rate is taken to be 1.65. The appropriate structures DBR is usually held near 0.91. These were originally calculated by Hulten and Wykoff (1981) in their landmark study, by averaging $d \times \sqsubset$ across six equipment (*versus* two structures) types for which they had obtained independent estimates of geometric depreciation and which were compatible with BEA's asset categorizations of that time.[4] In the years since, the two DBR estimates have become canonical: 24 out of 38 types of Private Nonresidential Equipment net stocks listed by the BEA in its compendium of the country's fixed assets (BEA, 2003) are tabulated with 1.65 as the DBR, as are 61 of 76 Government Nonresidential Equipment types and 9 of 12 Consumer Durables types. Further, 8 of 9 Private Residential types and all 10 Government Structures types are tabulated using a declining-balance rate of .91.[5] In every case, the declining balance rate was not derived from data or a strong theory, but only inherited from one of the two ancestral averages. A reexamination of declining balance rates and what they might imply is long overdue.

We'll approach the problem indirectly, by revisiting the tired complaint, "Depreciation can't possibly be geometric, because many assets do not lose the largest portion of their value when they are

---

[3] The cohort includes all its members, even those that have been retired, which are formally valued at zero.

[4] See Hulten and Wykoff (1981), pp. 81-125. The DBR computation is described on p. 94, and the use of old 1942-edition *Bulletin F* average service lives (reduced 15 percent) is explained on p. 100.

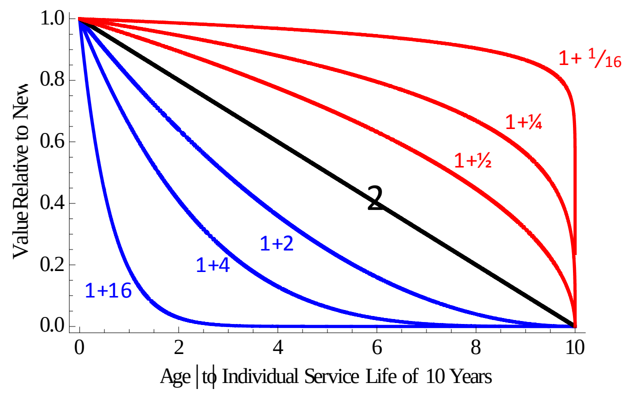
[5] Just one Private Nonresidential Structure type is tabulated with a declining-balance rate of .91, but DBRs for the other 24 are close, ranging between .8892 and .9747.

brand-new, and because no asset lasts forever." The complaint depends on the fallacy of composition. Consider a simple model of individual-level depreciation given by a resale-price profile of the form:

$$\theta_v(s,L_j) = (1 - s/L_j)^{v-1} \qquad \textit{for } 0 \le s < L_j, \textit{ or else } 0 \qquad (1.6)$$

where *s* stands for an individual's age and L_j its service life.[6] This is just a powered-up modification of the well-known straightline form. The parameter *v* is the same across individuals and is meant to indicate some commonality in how individual members of the same asset type lose value. Depending on *v*, a plot of relative-to-new resale values descends from 1 (full new price) to 0 either quickly at first and then slowly (for $v > 2$), or vice versa (for $1 < v < 2$), or in equal increments ($v = 2$).

Figure 1: Hypothetical v-Type Individual-Level Resale-Price Profiles



We shouldn't view the curves as equally likely; in fact, the red curves near the top of the graph are distinctly implausible (about which more later). Rather, the main idea of the graph is that equation (1.6) embraces a wide range of shapes, so it may approximate actual individual-level resale-price profiles fairly well, even though no individual depreciates in a geometric manner.

As an analytical demonstration of this last point, consider individual *j*'s age-*s* depreciation rate:

$$\delta_j = -\partial\theta_v(s,L_j)/\partial s \,/\, \theta_v(s,L_j) = (v-1)/(L_j - s) > 0 \text{ for } v > 1 \text{ and } 0 \le s < L_j \qquad (1.7)$$

This is constant neither across different individuals (i.e., different L_j) nor across ages for the same individual, which rules out geometric depreciation. Still, the expression at age 0 — $\delta_i(0) = (v-1)/L_j$ — already resembles (1.4) above, so there is hope to continue.

To press on, embed (1.6) in an expectational framework, so even if individuals do not depreciate geometrically, a cohort-average of them might. The problem then becomes one of finding a weighting function, $f(L)$, to satisfy:

---

[6] I assume against good sense that each individual's *L* is known, which is certainly false when the individual is new but maybe not so bad when it is old. Another working paper in this series examines the reduction in individual uncertainty that comes with improved knowledge of the individual's position in an aging cohort. At any rate, the aggregation exercise (which comes soon) mitigates the objection.

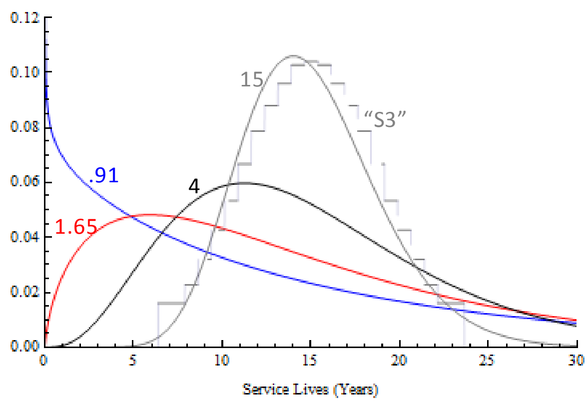
$$\Theta(s) = \int_s^\infty \theta_v(s,L)\, f(L)\, \mathrm{d}L \qquad (1.8)$$

where $\Theta(s)$ is from (1.3) and $\theta_v(s,L)$ is from (1.6). Formally, this is a *linear Volterra integral equation of the first kind.* In this case, the problem has an analytical solution, which is a Gamma probability density function (PDF) of service-lives:

$$f(L) = \delta^v L^{v-1} e^{-\delta L} / \Gamma(v) \qquad \textit{for } L \geq 0, \textit{ or else } 0 \qquad (1.9)$$

where $\delta>0$ is the *rate* parameter and $v>0$ is the *shape* parameter.[7] (Appendix 1 verifies the solution.) Figure 2 plots several Gamma densities, each with a mean of 15 years, for suggestive values of v:

Figure 2: Various 15-Year Gamma Densities



Notes on Figure 2: Only v's are listed; to find δ, divide v by 15. "S3" adapts to the continuous domain the modified Winfrey S3 discrete distribution BEA formerly used for weighting many individual nonresidential assets (though they were individually depreciated in straightline fashion — i.e., v=2).

The point here, akin to Figure 1, is the wide range of shapes that the 2-parameter Gamma PDF can capture. Most of them bulk left and trail right, in rough agreement with many survival studies, though one is almost symmetric (and relatively tight). The mean of the Gamma distribution is another weighted average, where now the PDF weights individual L's instead of individual depreciation profiles:

$$v/\delta = \int_0^\infty L\, \delta^v L^{v-1} e^{-\delta L} / \Gamma(v)\, \mathrm{d}L \qquad (1.10)$$

Setting this left-side result equal to the sample-average $\bar{L}$ unmasks v as the continuous-time form of the declining-balance rate:

$$v/\delta = \bar{L} \qquad (1.11)$$

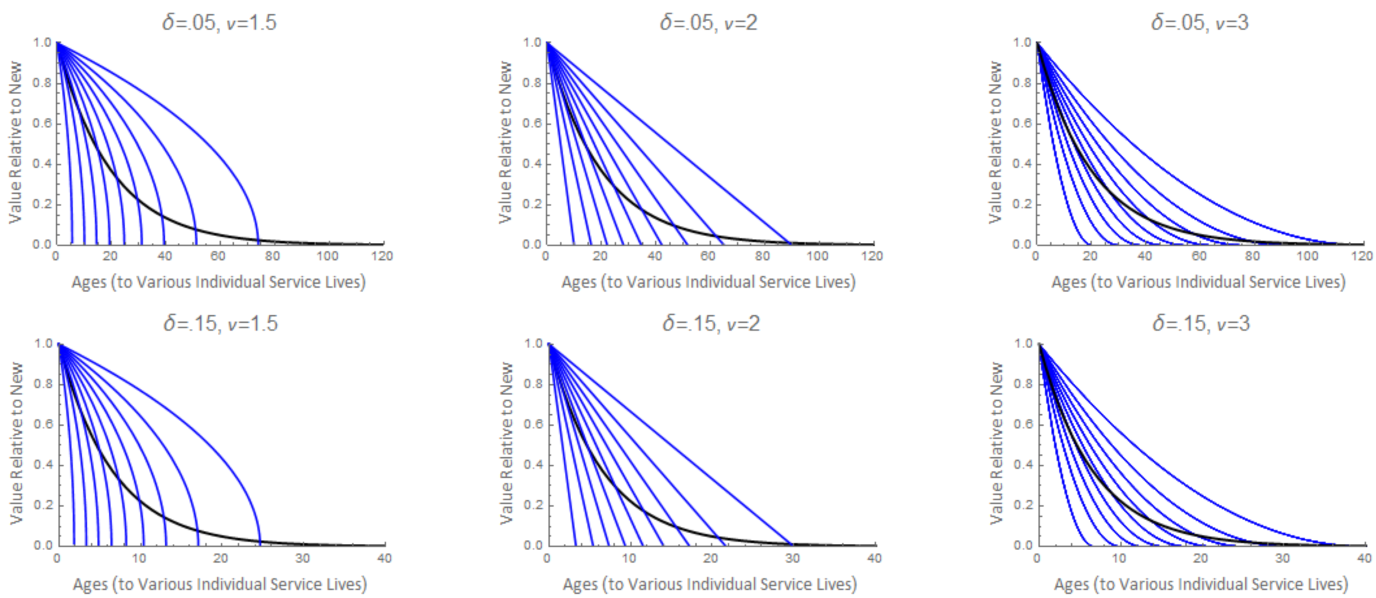
We have shown that v not only captures the persistence of an individual's value within its service life, but also the shape of the retirement distribution, so the declining balance rate has both individual- and cohort-level meaning. Moreover, since the variance of the Gamma distribution is v/δ^2, it is easy to

[7] The Gamma function $\Gamma(v)$ extends the factorial $(v-1)!$ to non-integer v.

estimate the depreciation rate as the ratio of the mean to the variance, and the declining balance rate as the ratio of the squared mean to the variance. In other words, to the extent that (1.6) adequately represents an individual asset's depreciation, all the parameters of a cohort's stipulated geometric depreciation can be estimated from the retirement patterns of the individual members. So capital accountants would do as well to monitor scrapyards as used-asset prices. (We'll examine in section 2 whether (1.6) is adequate for individuals, and in section 4 the sort of depreciation that happens to the cohort as a whole instead of differentially across individuals.)

A graphical demonstration of two geometric cohort-level resale price profiles (1.3), and three ways to compile them from individual-level profiles (1.6), follows.

Figure 3: Two Geometric Cohort Resale-Price Profiles Constructed from Six Densities of Three Types of Individual Resale-Price Profiles



Notes on Figure 3: The three plots in the first row each present a cohort-level resale-price profile (in thick black) that declines at 4.88 percent a year (equivalent to δ= .05), while the three in the second row each show one that declines at 13.93 percent a year (δ=.15). The nine blue lines in each plot are individual-level resale-price profiles per form (1.6), with service lives set to the expected values of the nine order-statistics drawn from nine-observation Gamma-distributed samples that correspond to the cohort-level geometric resale-price profiles. Individual-level resale-price profiles decline in a downwardly concave manner in the left two plots (v=1.5), in straightline fashion in the middle two (v=2), and in a downwardly convex manner in the right two (v=3).

2. Q: What's Not to Like? A: $v < 2$

Having rationalized capital accountants' longstanding practice in terms of a flexible Gamma distribution of service lives, and having provided a defensible, calculable declining balance rate, we can almost dispense with individual profiles of form (1.6). After all, capital accountants deal in asset cohorts, not individuals, and individual profiles are almost never observed. Their main use above was to provide plausible objects for the Gamma PDF to weight in the construction of the cohort geometric depreciation profile. We now re-examine their plausibility.

Each individual relative-to-new price profile may be thought of as the present discounted value of the future stream of rents (normalized by the new purchase price), so it may be strange not to see a (constant, say) interest or discount rate in expression (1.6), i.e.:

$$(1 - s/L)^{v-1} = \int_s^L e^{-r(u-s)}\, \mathrm{R}_v(u,L)\, \mathrm{d}u \tag{2.1}$$

where u indexes *future* ages (so $s \le u \le \mathrm{L}$), r is the constant own-discount rate, and the unspecified rent function $\mathrm{R}_v(u,L)$ depends both upon its future ages and, it would seem, upon r (to cancel somehow with r in the discounter $e^{-r(u-s)}$, so that r never makes it out of the integral).[8]

To find $\mathrm{R}_v(u,L)$ in (2.1) at age u=s, apply the well-known rental-price transformation:[9]

$$\mathrm{R}_v(s,L) = \left[r - \frac{\partial}{\partial s}\right](1 - s/L)^{v-1} = \left(r + \frac{v-1}{L}\right) \times \left(1 - \frac{s}{L}\right)^{v-1} \frac{r + \frac{v-1}{L-s}}{r + \frac{v-1}{L}} \tag{2.2}$$

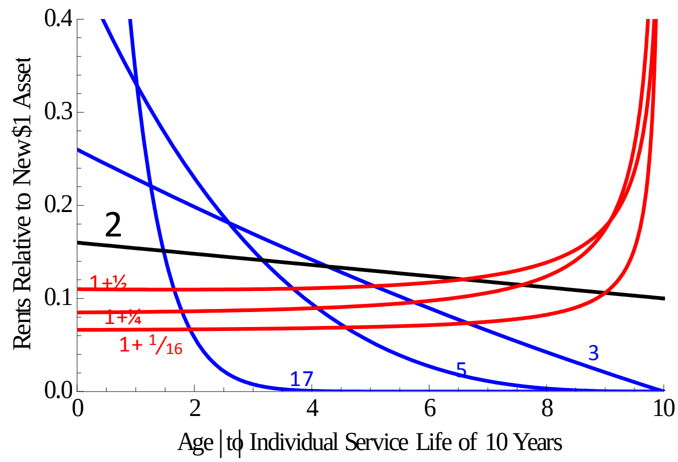
The result is adequate for v above 2 — i.e. corresponding to (blue) individual price profiles in the lower-left half of Figure 1 — but leaves rents on the table in the amount of 1/L for v=2 as the individual's age attains the service life. Even worse, for v between 1 and 2 — i.e. which correspond to (red) individual price profiles in the upper-right half of Figure 1 — the individual rent function *climbs without limit* as the age approaches the service life. The implication is that such assets become astonishingly productive just before their retirements. For v below 1, the rent function turns negative, while the individual price profile itself increases without limit as age approaches the service life. The case of v=1 represents the post-Napoleonic British consol bond, which sells for 1 and pays r forever. Figure 4 on the next page shows individual-level rental profiles for the different values of v that correspond to the relative-to-new price profiles of Figure 1, for r = .06. It would seem that fully half the space of individual relative-to-new price profiles — i.e., where $v \le 2$ — is invalidated by the behavior of its implied rental profiles. As a

[8] This makes the math work, but the economics do not require it. One might set R(u,L) independent of r, forcing the whole interest-rate response onto the resale-price profile, or allow different but compatible degrees of interest sensitivity by R(u,L) and the price profile. See equations (2.3)-(2.6), below.

[9] The "operator notation" $[r - \partial/\partial s]$ is intended to save space. It means: "multiply by r, then subtract the derivative with respect to s, of the expression that immediately follows." The right-side expression is factorized into the individual-level user-cost (i.e., the rental-price at age 0, in blue) for an asset that sells for $1 new, times the individual-level age-efficiency profile (in black). For the reader expecting the well-known $(r + \delta)$ per dollar as the user-cost, *cf.* (1.7) and the discussion following, above, where $\delta(0) = (v-1)/\mathrm{L}$.

consequence, a service-life histogram that suggests a Gamma distribution with a shape parameter *v* of 2 or below is, at this stage, problematic. So far, that is most of BEA's capital stock.[10]

Figure 4: Hypothetical v-Type Individual-Level Rental-Price Profiles



Notes on Figure 4: Each rental-price profile shown here corresponds to the *resale*-price profile shown in Figure 1. It should be plain that only for $v \leq 2$ is form (2.2), and by extension (1.6), safe to use. Serviceability may decrease as an individual ages, but it should not improve miraculously as the service-life nears. Other families will allow concave-downward rents.

There is less room to wriggle out of this problem than we might like. The constraint that individual price profiles weighted by a service-life density integrate *exactly* to a cohort-level geometric price profile is severe, and the Gamma form is already flexible enough to capture realistic features of retirement data. So keep the Gamma family of service-life densities, as these yield both δ and *v*, but find individual price profiles that *approximate* (1.6) *yet still* have well-behaved implied-rent functions. Two candidates here are the "*a*-form" and "*b*-form" relative-to-new resale-price profiles:

$$\theta_a(s,L) = \frac{a\left(1-e^{r(s-L)}\right)-rL\left(1-e^{a(s/L-1)}\right)}{a\left(1-e^{-rL}\right)-rL\left(1-e^{-a}\right)} \qquad (a \text{ Real}) \qquad \textit{for } 0 \leq s < L, \textit{ or else } 0 \qquad (2.3)$$

and

$$\theta_b(s,L) = \left(\frac{e^{\frac{rs}{1+b}}-e^{\frac{rL}{1+b}}}{1-e^{\frac{rL}{1+b}}}\right)^{1+b} \qquad (b \geq 0) \qquad \textit{for } 0 \leq s < L, \textit{ or else } 0 \qquad (2.4)$$

These have, by the rental-price transformation, the respective rental functions:[11]

---

[10] For DBR=1.65, an asset would need $d > .3283$ — i.e., $\bar{L} < 5.03$ years — to attain $v > 2$.

[11] As with (2.2), the rental price profiles (2.5) and (2.6) factorize into age-0 rental prices termed "user costs" (in blue) for an asset that sells for $1 new, and "age-efficiency profiles" (in black), so the age-efficiency profiles are definable as *ratios* of age-*s* rental prices to user costs.

$$R_a(s,L) = \frac{r(a-rL)}{a(1-e^{-rL})/(1-e^{-a})-rL}\left(\frac{e^{a\,s/L}-e^{a}}{1-e^{a}}\right) \qquad (a \text{ Real}) \qquad \textit{for } 0 \le s < L, \textit{ or else } 0 \qquad (2.5)$$

and

$$R_b(s,L) = \frac{r}{1-e^{-\frac{rL}{1+b}}}\left(\frac{e^{\frac{rs}{1+b}}-e^{\frac{rL}{1+b}}}{1-e^{\frac{rL}{1+b}}}\right)^{b} \qquad (b \ge 0) \qquad \textit{for } 0 \le s < L, \textit{ or else } 0 \qquad (2.6)$$

Despite their formidable appearances, neither form was pulled out of a hat. (See Appendix 2.) Both depend on a curvature parameter (i.e., a or b, neither of which easily reduces to v) and an own/real discount rate r. Both imply user-costs — i.e., the blue parts of (2.5) and (2.6) — that remain positive *even for negative* r, which can be useful for measurement in some macroeconomic contexts.[12]

Further, both forms are *reasonable*:

- Non-increasing resale-price profiles *and* efficiency profiles — i.e., the black parts of (2.5) and (2.6) — as age increases from 0 to L, with substantial efficiency flows throughout an individual career (i.e., no retirement-in-place)
- Straightforward derivative (i.e., user-cost) and integral (i.e., net-present value) transformations back-and-forth between the price and rental forms, without need for higher functions
- Flexible approximation of unknown but smooth individual price and efficiency patterns
- No premature retirements

None of the four suggested features is air-tight. Practical experience suggests some assets' service-flows do increase for a time before tapering off (e.g., young dairy cows), yet for many assets, any increase is really about rookie operators' skills, not the assets themselves. The prohibition against higher math is for the benefit of human economists, not their computers. The hyperbolic age-efficiency profiles of the U.S. Bureau of Labor Statistics (BLS) would be unreasonable by this second criterion, even though its range of shapes is not much different from the shapes available to forms (2.5) and (2.6). "Flexible approximation" aspires to something more technical. The a-form price profile (2.3), for instance, solves the constant-coefficient inhomogeneous second-order linear differential equation:

$$\theta_a''(s) = c_{oo} + c_o\,\theta_a(s) + c_1\,\theta_a'(s) \qquad (2.7)$$

and so has appeal as an approximation akin to a second-order Taylor series.[13] For its part, the rate of change of the b-form profile (2.4) — i.e., $G_b(s) = \theta_b'(s)/\theta_b(s)$ — solves the logistic differential equation:

$$G_b'(s) = d_1\,G_b(s)\,(d_2 - G_b(s)), \qquad (2.8)$$

---

[12] So a service-life -frequency weighted average of the user-costs of individuals of form (2.5) or (2.6) would stay positive as well. (N.B.: User-costs are relative to a \$1 new purchase price.) Note that in the individual user-cost of form (2.2) *only* — i.e., $r + (v-1)/L$ — the discount and depreciation rates are separable. The cohort geometric form, with user-cost $r+\delta$, inherits that separability, since $E_L[(v-1)/L] = \delta$, where the expectation is taken over the Gamma density of service-lives.

[13] The coefficients are: $c_{oo} = \frac{r(a-rL)}{a(1-e^{-rL})/(1-e^{-a})-rL}\frac{a/(1-e^{-a})}{L}$, $c_o = -\frac{ar}{L}$, and $c_1 = r + \frac{a}{L}$. For its part, the a-form rental profile (2.5) solves a constant-coefficient inhomogeneous *first*-order linear differential equation, so (2.3) is arguably a better approximant than (2.5), even though the two forms are dual to each other.

which is applied widely to model dynamic processes.[14] Finally, the ban on premature retirements — that is, against removing assets from service while there is still a shred of value in them (equivalently, the assumption of a zero scrap value) — is arguably *not* reasonable, but it does conform to national accounting conventions. (I hope in future work to allow positive scrap values to address early, obsolescence-induced retirements.)

Given that the exact resale-price form (1.6) is not reasonable, by fault of its implied rental form (2.2), can we instead find values of *a* (or *b*) and *r* in θ(s,*r*) and R(s,*r*) such that:

$$e^{-\delta s} \approx \int_s^\infty \theta_{a\ or\ b}(s,L)\ \frac{\delta^{v} L^{v-1}}{e^{\delta L}\Gamma(v)} dL \tag{2.9}$$

and/or:

$$e^{-\delta s} \approx \int_s^\infty R_{a\ or\ b}(s,L)\ \frac{\delta^{v} L^{v-1}}{e^{\delta L}\Gamma(v)} dL/(r+\delta) \tag{2.10}$$

for all relevant ages *s*, given some preselected δ and *v* — in particular, for *v* in the troublesome region between 1 and 2? If so, then the method of fitting a collection of service lives to a Gamma distribution, to find out everything needed for geometric depreciation, is approximately back in business.

To test this, I looked up BEA's longest- and shortest- lived equipment asset types with DBRs of 1.65 in Table C of *Fixed Assets and Consumer Durable Goods* (FACD 2003). The longest lived — Electrical transmission, distribution, and industrial apparatus; Tennessee Valley Power Authority equipment; Bonneville Power Authority equipment; and Switchgear and switchboard equipment — all have 33-year average service lives, implying 5 percent annual depreciation rates of the form $(1–.05)^s$. Compatibility with the Gamma distribution necessitates the transformation to $\delta = -ln(1-d) = .0512933$, which in turn implies $v = \text{DBR}\times\delta/d = 1.69268$. The shortest-lived type, Ophthalmic products and orthopedic appliances, has a 6-year average service life, implying a 27.5 percent *d*-form annual depreciation rate and so δ=0.321584 and *v*=1.9295. Most other equipment types have average service lives and depreciation rates between these two cases.[15] For *v* and δ and choices of *r*, I selected the best *a* (or *b*) to minimize a 50:50 average of summed squared residuals between the left and right sides of (2.9) and (2.10) over 100 evenly-spaced ages between 0 and the age at which the cohort-average price was only 1 percent of its original value.[16] Graphical summaries of the ranges of fitted point-values are in Figure 5. For either form, and for the range of equipment types between 6-year (red) and 33-year (blue) average service lives, the best fits generally imply rental profiles that are downwardly concave (i.e., $a > 0$ for any *r*, and $b \le 1$ for $r > 0$.[17]) In the *a*-type approximations, summed squared errors are minimized near *r*=.06 for assets with 6-year average service lives but *r*=.48 for assets with 33-year average service lives. The *b*-

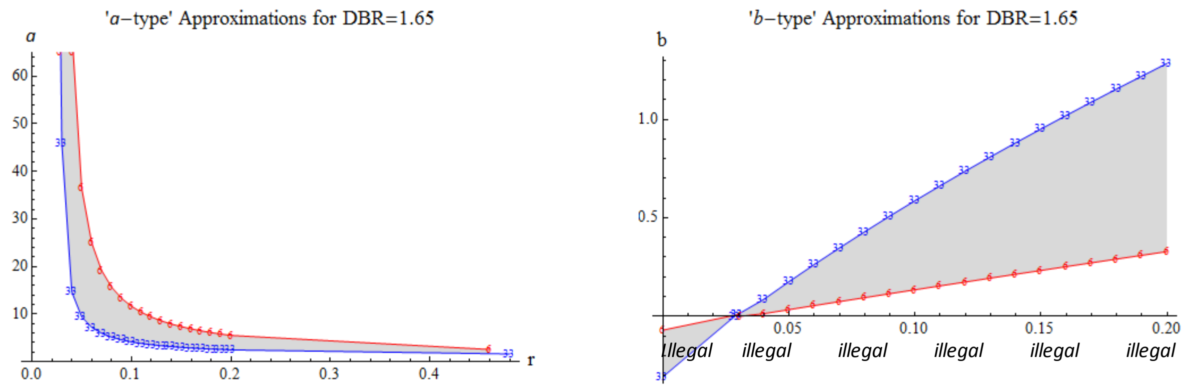
[14] The coefficients are: $d_1 = \frac{1}{1+b}$ and $d_2 = r$. This is less appealing than a connection to a Taylor series, yet logistic models of dynamic processes are flexible and in broad use. Moreover, the rate of change of the *b*-form rental profile (2.6) solves a related logistic differential equation, and so is as good an approximant as the price form.

[15] The same transformation of $d\rightarrow\delta$ and DBR→*v* moves Prepackaged software, with $\bar{L}$=3 (so *d*=0.55), to (δ,*v*) = (0.798508, 2.39552), and Custom and Own-account software, with $\bar{L}$=5 (*d*=0.33), to (δ,*v*) = (0.400478, 2.00239). As all three software types wind up with shape parameters above 2, forms (1.6) and (2.2) would still work exactly.

[16] I carried out the exercises across a range of *r*-values, instead of seeking the best (*a*,*r*) or (*b*,*r*) pair. Choosing two parameters to replace one (i.e., *v*) is an ill-posed problem — that is, broad swathes of (*a*,*r*) and (*b*,*r*) pairs would be adequate for (2.9) and (2.10), even if a particular pair of (*a*,*r*) or (*b*,*r*) values is numerically (and dubiously) "best."

[17] Even for $b > 1$ (and positive *r*), the rental profile is downwardly concave at ages below $L-(1+b)ln(b)/r$.

Figure 5: Use $\theta_a(s,r)$ or $\theta_b(s,r)$ to Approximate $\theta_v(s,r)$, and $R_a(s,r)$ or $R_b(s,r)$ to Approximate $R_v(s,r)$, in Cohort Aggregations of 6-Year and 33-Year Equipment Assets



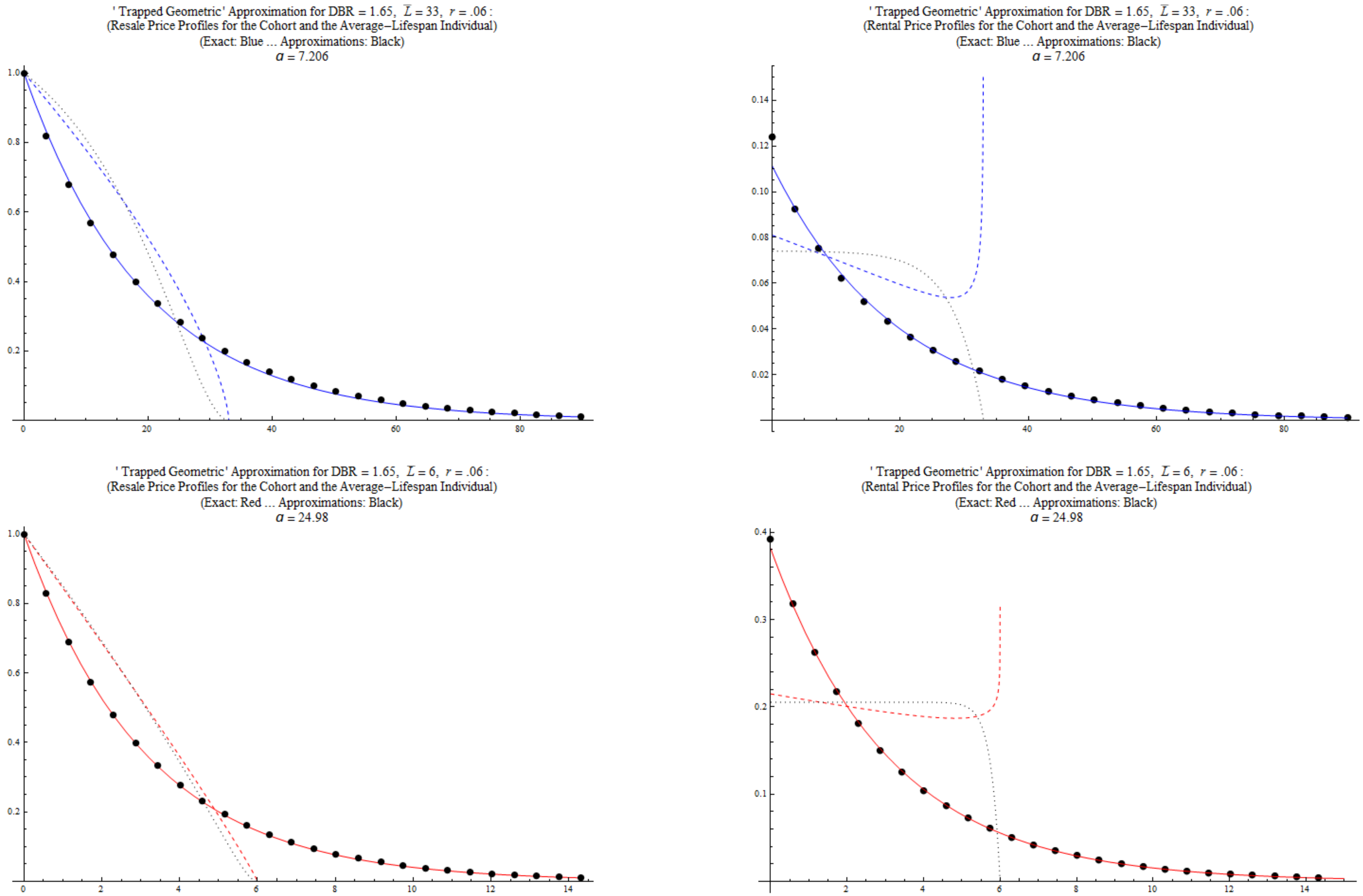
Notes on Figure 5: The L-shaped red schedule in the left plot shows the best choices of curvature parameter *a*, given *r*, to minimize squared differences between the left and right sides of (2.9) and (2.10) for 6-year equipment assets, using (2.3) instead of (1.6) and (2.5) instead of (2.2). The L-shaped blue schedule does the same for 33-year equipment assets. The two upward-sloping schedules in the right plot represent the best choices of b given *r*, using (2.4) instead of (1.6) and (2.6) instead of (2.2). Illegal $b<0$ drives (2.6) to increase sharply as $s \rightarrow L$, just as $v<2$ drives (2.2).

type approximations are quite poor for high *r*, always preferring smaller *r* (and *b*), down to the one-hoss shay limit of $b=0$ (for *r* near .03), and would in fact coincide with forms (1.6) / (2.2) at $r=0$, where *b* would (illegally) equal $v-2$. For all nonnegative *b*, however, the *a*-type approximations are tighter.

For a visual sense of the fits at a conventional own-interest rate ($r = .06$), Figure 6, below, presents geometric cohort resale-price and rental profiles for asset types with 33- and 6-year average service lives, together with approximations to the cohorts constructed as Gamma-weighted averages of *a*-type individual resale and rental profiles. The Figure is structured as four panels. Resale profiles occupy the left two panels and rental profiles the right two, while 33-year assets (in blue) are shown in the top two panels and 6-year assets (in red) in the bottom two. At the cohort level, exact geometric forms are solid lines, while the approximations track the heavy black dots (•). At the individual level, exact *v*-type resale and rental profiles for 33- or 6-year service lives are given as blue or red dashes (- - -), respectively, while the *a*-approximations are shown as light black dots (····). Figure 7 goes through the same exercises, based on *b*-type individual-level resale and rental profiles.

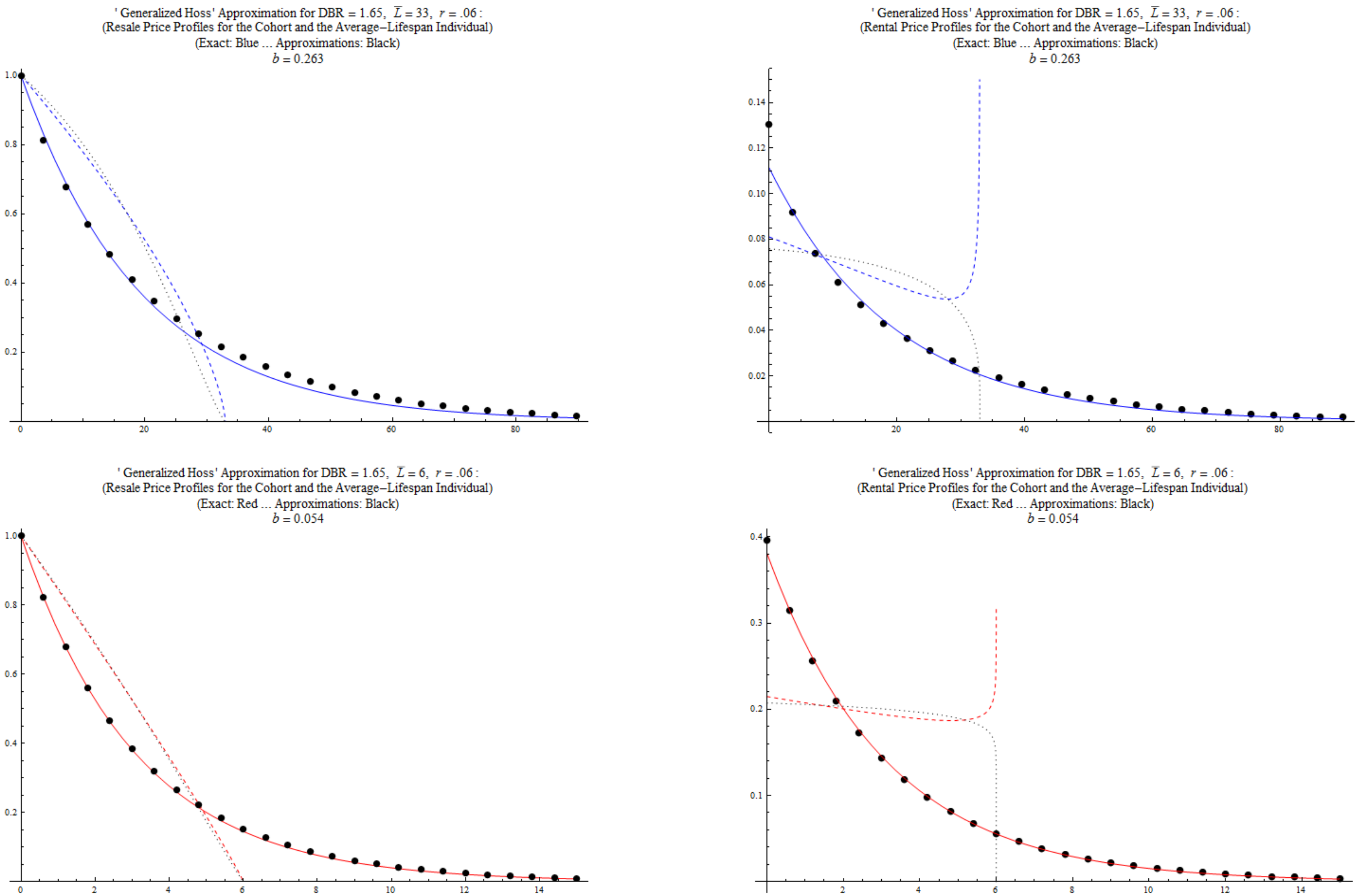
Both the *a*- and *b*-type approximations match the features of the 6-year mean-lifespan cohort better than those of the 33-year cohort, though the discrepancies — i.e., too-high tail values for the resale schedule, and a too-high starting value for the rental schedule — are plainer on the *b* page. Both *a*- and *b*-type approximations track the individual resale schedule well enough, and both accomplish their intended missions, replacing implausible *v*-type individual rental schedules. All in all, the approximations are adequate to rescue the interpretation of depreciation and declining-balance rates in terms of the parameters of a Gamma service-life distribution, even for values of *v* between 1 and 2. On that score, BEA's equipment assets, with *v* near 1.65, are safe.

Figure 6: How Well Does θ_a (2.3) Replace θ_v (1.6), and How Well Does R_a (2.5) Replace and Correct R_v (2.2), in Comparisons across Individuals and Cohorts?



Notes on Figure 6: The two left plots compare resale-price profiles, and the two right plots compare rental-price profiles, at the (geometric) cohort level (solid lines and big-dot approximations) and for an individual with the mean service life (light dashed and dotted schedules). The top two plots represent $\bar{L}$=33 years, the bottom two $\bar{L}$= 6 years.

Figure 7: How Well Does θ_b (2.4) Replace θ_v (1.6), and How Well Does R_b (2.6) Replace and Correct R_v (2.2), in Comparisons across Individuals and Cohorts?



Notes on Figure 7: The two left plots compare <u>resale</u>-price profiles, and the two right plots compare <u>rental</u>-price profiles, at the (geometric) cohort level (solid lines and big-dot approximations) and for an individual with the mean service life (light dashed and dotted schedules). The top two plots represent $\bar{L}$=33 years, the bottom two $\bar{L}$= 6 years.

On the other hand, the pronounced concavity of the individual-level *a*- and *b*-type rental profiles suggests the approximations are near the edges of their feasibility. First, there is little room for obsolescence, which (for a single unexpected occurrence) may be modeled as a common factor $0<(1/B)<1$ multiplying all *v*-, *a*-, or *b*- resale and rental forms from the common date that an external force makes them less valuable.[18] (So "resetting" an individual's age to zero subsequent to that damning date would only restore its relative value to $1/B$ not 1.) A *continuous* (and presumably expected) constant obsolescence *rate* of $-lnB$, incurred every instant since an individual's installation as the (then) frontier asset, would multiply resale and rental forms by $e^{-lnB\,s}$ and increase the effective interest rate from r only to $r + lnB$. The two effects work in opposite directions, but the multiplicative effect dominates, pushing down the individual profiles. By contrast, one of the great simplifications of the cohort geometric form is that δ captures both aging and obsolescence. Supposing that depreciation of our 6- and 33-year asset types already includes a large obsolescence component, which the individual *v*-, *a*-, and *b*- forms do not explicitly model, implies that the individual-level curves are actually *products* of realized-obsolescence and age effects (the latter elevated slightly by higher effective interest rates). The former effect is holding the individual curves down, so the unobserved curve that traces out the latter effect must be even higher than what is drawn. Yet what's drawn already grazes the one-hoss-shay ceiling.[19] Section 4 will discuss obsolescence, whether sudden or steady, further.

Second, both the *a*- and *b*-type resale-price approximations match v=1 only at the one-hoss shay limit (i.e., $a\to\infty$ or b=0) with no-tomorrow discounting ($r\to\infty$).[20] They cannot feasibly (in the *a*-case) or legally (in the *b*-case) match any $v < 1$. Yet every one of BEA's structures assets has $v < 1$. A quick δ-preserving fix would increase both v and Γ in the same proportion, though I will consider later whether BEA's depreciation-rate estimates for structures are in fact too slow.

In terms of comparability with other agencies, the individual *a*- and *b*-type rental profiles drawn in Figures 6 and 7 are more concave than the normalized rental profiles that BLS uses for its individual-level age-efficiency profiles (which, like the *v*-, *a*-, and *b*- forms laid out here, make no explicit provision for realized obsolescence). Yet BLS' cohort-level productive-stock profiles by asset-type are less convex than BEA's geometric cohort depreciation profiles (which, being geometric, serve for both wealth and productive stocks). It would seem, then, that the empirical nub of methodological disagreements between the two agencies over the proper measurement of depreciation can be cast as differences in how individual-level profiles are weighted.

---

[18] If one could compare the sale-price of a new asset of the very latest vintage to the sale-price at the same date of an asset from a *passé* vintage, where the second asset has been costlessly made new again, the ratio of the first price to the second would be *B*, the degree by which the first, "frontier" asset is better. See section 4, below.

[19] Recall that a one-hoss shay individual-level rental-price profile is entirely horizontal until the asset's age attains its service-life, at which all further rents drop to zero.

[20] Type-*b* approximations require only $r\to\infty$ to match the v=1 limit irrespective of the value of *b*, though smaller-*b* approximations approach that limit sooner.

3. Seeking Gamma

Next, consider a reverse problem. *Given* a targeted cohort-level geometric resale-price function such as (1.3) and individual resale profiles such as (1.6) or (2.3) or (2.4), all with preset parameters, what weights are needed to rationalize the former as an average of one of the latter? For form (1.6), we already know a Gamma probability density is the exact solution. What about for flexible forms (2.3) or (2.4)? These, recall, aspire to approximate any reasonable individual resale-price profile. If an implicit weighting function can be found to map (2.3) or (2.4) to (1.3), and if it is usefully *close* to a Gamma PDF, then we have grounds for thinking a Gamma distribution of lifespans is nearly *sufficient* for an asset-type cohort to depreciate in a stipulated geometric manner. If so, data on the distribution of (~zero-valued) service-lives *alone* would be enough to inform a good first guess of the parameters of geometric depreciation. A lack of used-asset prices would not be an impediment.

This can be tested. Return to (1.8) or (2.9), but rewrite the integral as a sum across a fine-grained range of potential service-lives; then do so across a commensurate fine-grained range of ages. For grain-size Δ, say, let $\theta(s,L_j)$ represent the relative-to-new price that an individual with idiosyncratic service-life L_j takes at age s, and let $w(L_j)$ be the relative frequency of L_j. Then we have:

$$
\begin{aligned}
&s=0: && e^{-0\delta} = 0\,w(0) + \theta(0,\Delta)\,w(\Delta) + \theta(0,2\Delta)\,w(2\Delta) + \theta(0,3\Delta)\,w(3\Delta) + \theta(0,4\Delta)\,w(4\Delta) + \theta(0,5\Delta)\,w(5\Delta) + \dots \\
&s=\Delta: && e^{-\delta\Delta} = 0\,w(0) + 0\,w(\Delta) + \theta(\Delta,2\Delta)\,w(2\Delta) + \theta(\Delta,3\Delta)\,w(3\Delta) + \theta(\Delta,4\Delta)\,w(4\Delta) + \theta(\Delta,5\Delta)\,w(5\Delta) + \dots \\
&s=2\Delta: && e^{-2\delta\Delta} = 0\,w(0) + 0\,w(\Delta) + 0\,w(2\Delta) + \theta(2\Delta,3\Delta)\,w(3\Delta) + \theta(2\Delta,4\Delta)\,w(4\Delta) + \theta(2\Delta,5\Delta)\,w(5\Delta) + \dots \\
&s=3\Delta: && e^{-3\delta\Delta} = 0\,w(0) + 0\,w(\Delta) + 0\,w(2\Delta) + 0\,w(3\Delta) + \theta(3\Delta,4\Delta)\,w(4\Delta) + \theta(3\Delta,5\Delta)\,w(5\Delta) + \dots \\
&s=4\Delta: && e^{-4\delta\Delta} = 0\,w(0) + 0\,w(\Delta) + 0\,w(2\Delta) + 0\,w(3\Delta) + 0\,w(4\Delta) + \theta(4\Delta,5\Delta)\,w(5\Delta) + \dots \\
&\dots
\end{aligned}
\tag{3.1}
$$

etc., up to some very large value L^{top} for the service life and an almost-as-large $s = L^{top}-\Delta$. Increasing age spells the demise of ever more members, as seen by the rightward spread of zero-valued individual prices. The unseen final line of the table would amount to $e^{-\delta\,(L^{top}-\Delta)\Delta} = \theta(L^{top}-\Delta,\ L^{top})\,w(L^{top})$. Setting $w(0)=0$ provisionally and noting all relative-to-new prices (including $e^{-\delta\,s}$) equal 1 at age 0, we see the first line forces lifespan weights to sum to 1.[21] Given preset values for δ and for the common parameters of the individual price functions (e.g., *a* or *b*, and *r*), the system may be back-solved from the final line up to the first. Dividing the solved-for weights by Δ and "connecting the dots" from L=0 through $L=L^{top}$ encloses an area of 1.

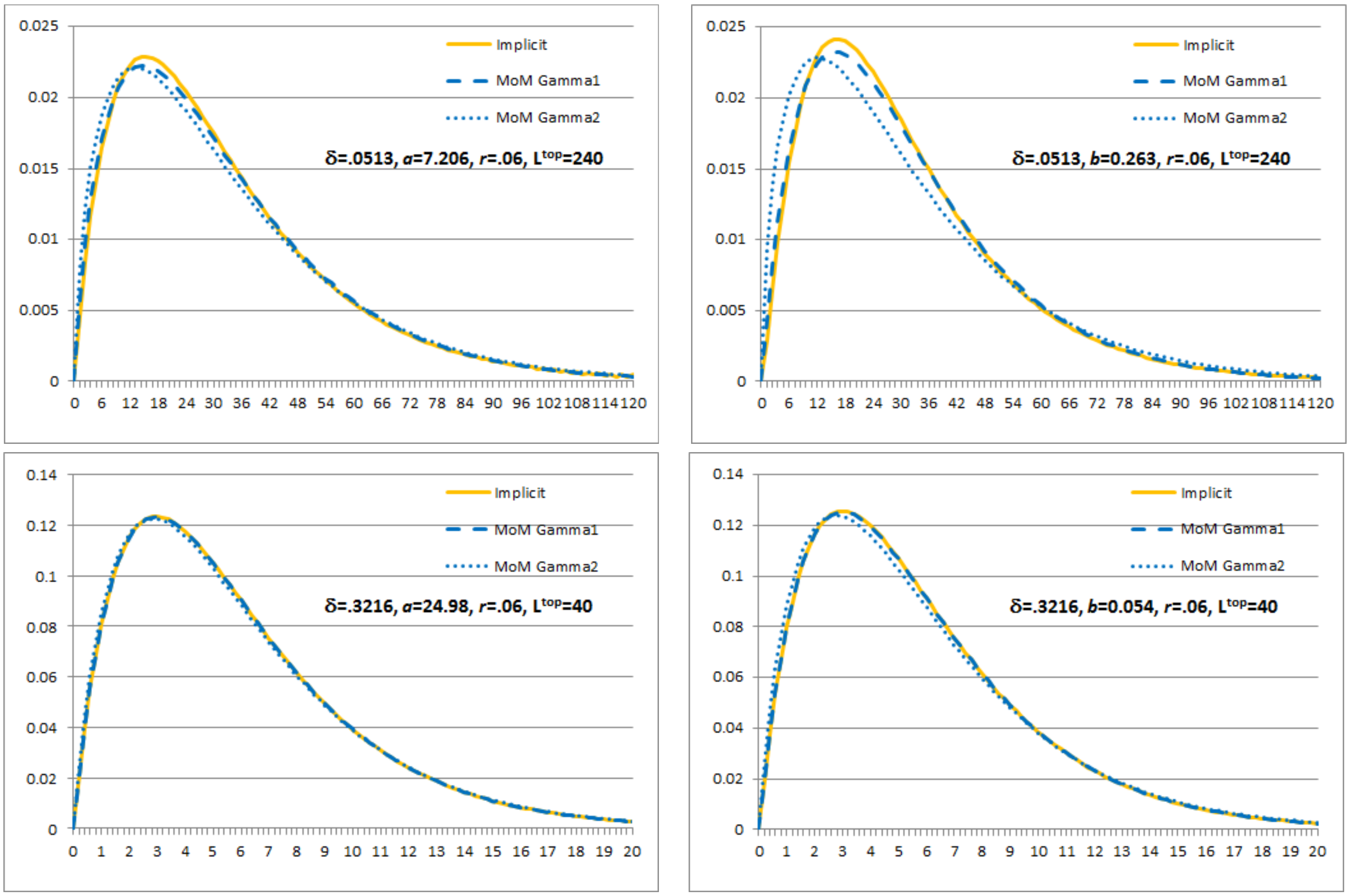
I carried out the just-described procedure in a large *Excel* workbook that is available to the interested reader. Along with the implicit numerical density, I included calculated two "nearby" Gamma densities — one (called "MoM1") matching the mean and variance of the implicit density, the other ("MoM2") matching the mean but constrained to the preset overall δ. Then I carried out several tests.

The first tests merely take the results of the 33-year specifications of Figures 6-7 (i.e., $a = 7.206$ or $b = 0.263$ for $r = 0.06$) and find the implicit distributions of service-lives to match the cohort geometric depreciation profile of rate $\delta = -ln(1 - 1.65/33) \approx .0513$. The implicit distributions have somewhat sharper

[21] Weights should all be nonnegative also, but I have not imposed this.

modes than the neighboring Gamma densities but are nonetheless quite similar.[22] Using results from the 6-year specifications of Figures 6-7 (i.e., a = 24.98 or b = 0.054 for $\delta = -ln(1 - 1.65/6)$ and r = 0.06), gives even better agreement between the implicit and Gamma densities. Figure 8 plots the four preliminary tests:

Figure 8: Confirm/Reverse the Approximations of Figures 6-7



Notes on Figure 8: In each plot, the gold line represents the implicit service-life distribution, the blue dashes (- - -) the Gamma density with the same first two moments, and the blue dots (····) the Gamma density with the same mean and preset δ. The top two plots treat $\bar{L}$=33 years and the bottom two treat $\bar{L}$ = 6 years. The left two plots show the implicit densities that best suit θ_a (2.3) and the right two the densities best for θ_b (2.4).

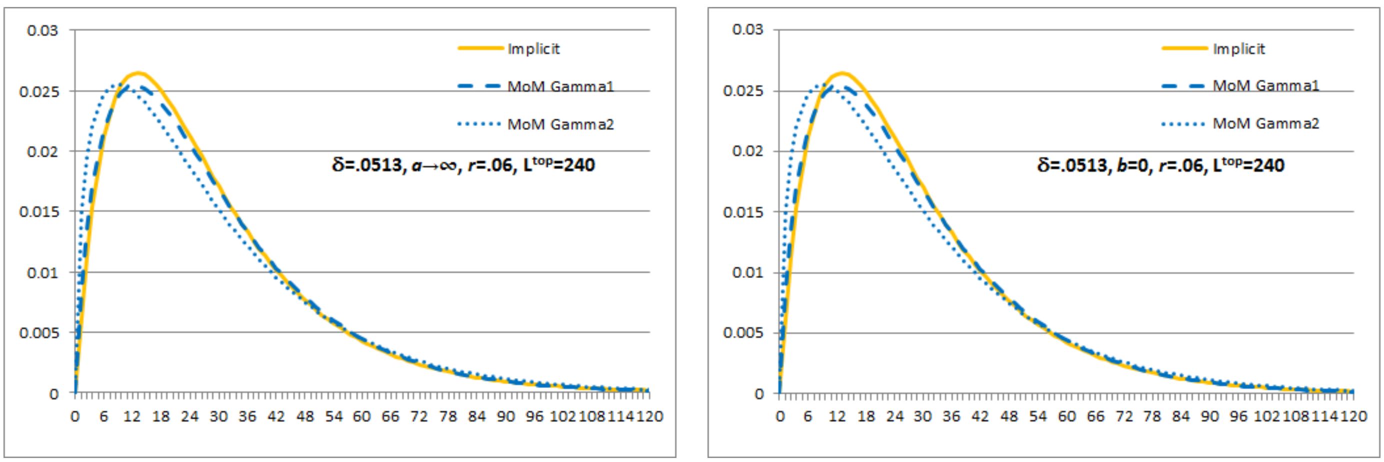
Such tests aren't onerous, but they do little besides confirm the earlier approximations. Nearly any (a,r) or (b,r) pair from the gray regions shown in Figure 5 would imply "near-Gamma" distributions of service lives — not just the conventional r = 0.06 "slice" of each region to which we've paid the most attention — because all such pairs were derived as approximations to individual profiles weighted by *exact* Gamma densities with parameters $\delta = -ln(1-1.65/\bar{L})$ and $v = \delta\bar{L}$. We need harder tests.

---

[22] The exact Gamma density for which a=7.206 or b=0.263 are approximate individual-level solutions, given r=0.06, had $\delta = -ln(1 - 1.65/33)$ and $v = \delta\times 33$. I haven't drawn it, but it lies between the two "MoM" curves, closer to "MoM2."

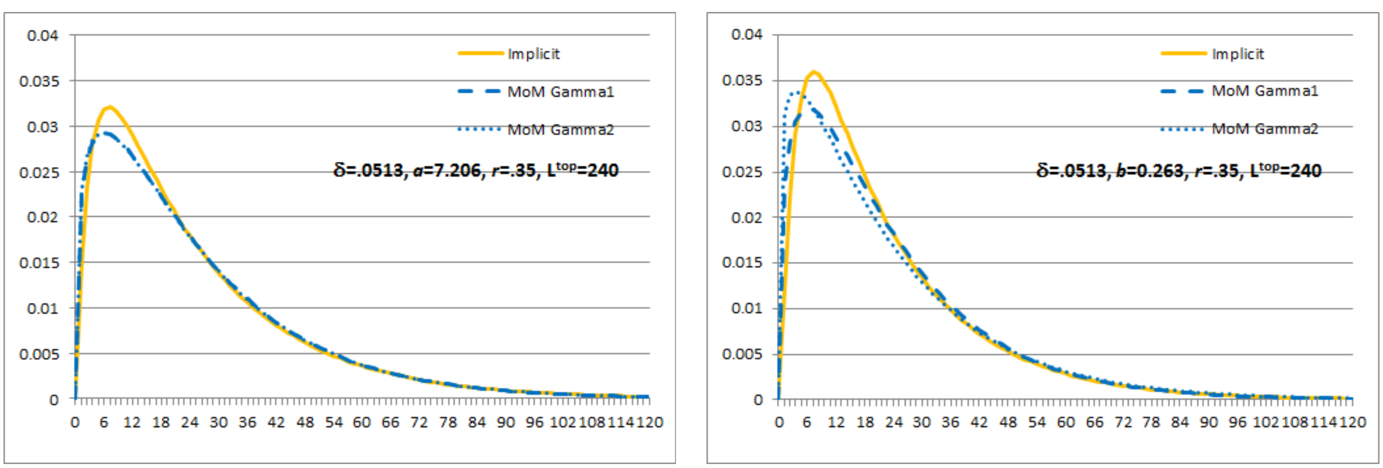
So instead consider service-life densities implied by "wrong" parameters: not merely other (a,r) or (b,r) points along the red "6-year" or blue "33-year" boundaries of the plots in Figure 5, but rather (a,r) or (b,r) points that are *off the boundaries* and *outside* Figure 5's gray areas. Some experimentation showed that for fixed values of δ and r, increasing a (i.e., starting from within the gray region of the left plot of Figure 5 and moving "north") or reducing b (i.e., starting from within the *legal* gray region of the right plot and moving south) — i.e., which amounts to moving individual curves toward the one-hoss shay case — is essentially harmless. A Gamma density with the same mean and variance as the implicit density is always "near" the implicit density. The same holds for increasing r, given δ and a (or b), although *very* high r (e.g., ~40%) make the implicit densities quite peaked. And increasing δ, given r and a (or b), always tightens gaps between the implicit densities and their Gamma partners. The next four Figures show representative departures from the {δ=.0513, a=7.206 or b=.263, r=.06} base case.

Figure 9: Drive Individual-Level Resale-Price Profiles toward One-Hoss Shay Limit



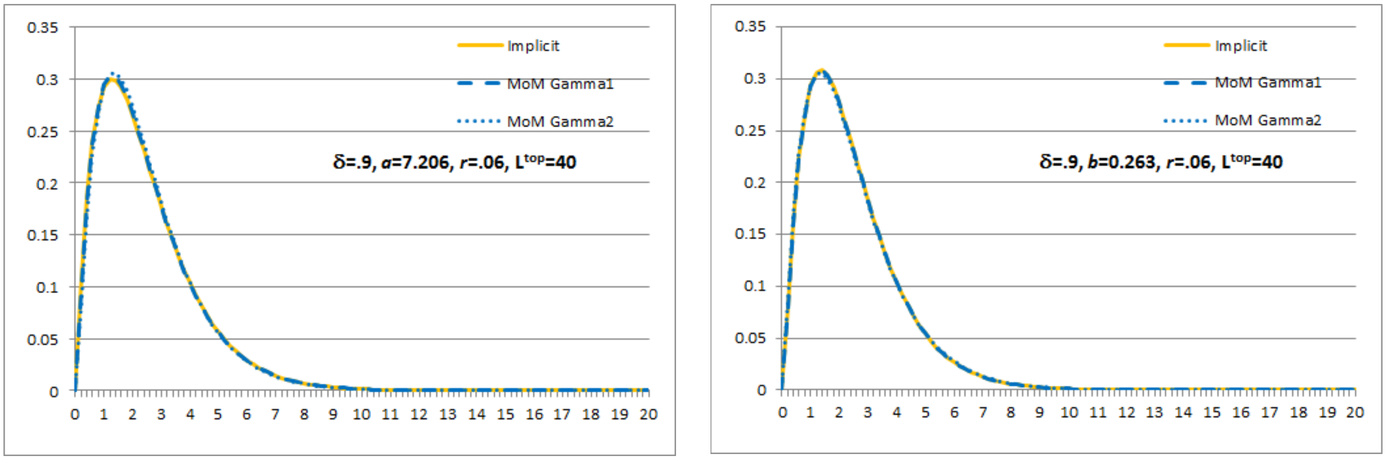
Notes on Figure 9: In the limit, as $a\to\infty$, (2.3) drives to the resale-price function for a one-hoss shay individual. The same is true for (2.4) as $b\to0$. The one-hoss shay θ is $(e^{rs} - e^{rL})/(1 - e^{rL})$. For a given r, the dual individual-level age-efficiency profile shows no efficiency loss until the asset is retired. Figure 9 differs little from the top two plots of Figure 8 because, at a=7.2 and b=.26, the individual-level profiles are already fairly close to one-hoss shay.

Figure 10: Increase the Own Rate of Return



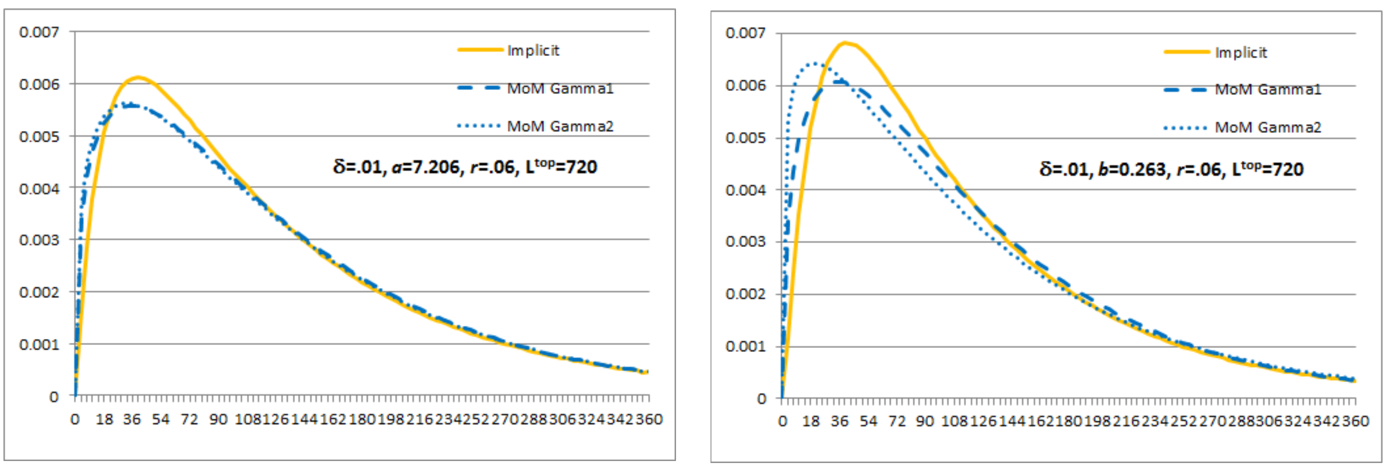
Notes on Figure 10: In the limit, as $r \to \infty$, resale-price functions converge on their age-efficiency duals. The effects are already apparent at r=.35. Age-efficiency profiles ride higher than resale-price profiles, so a given cohort-level resale-price profile can only be maintained if service-lives retract toward 0, so densities are very thin for large service-life values.

Figure 11: Speed Up Cohort Depreciation



Notes on Figure 11: Increasing δ to .9 (equivalent to 59 percent annual depreciation) severely retracts service lives. Implicit and Gamma densities visually coincide.

Figure 12: Slow Down Cohort Depreciation



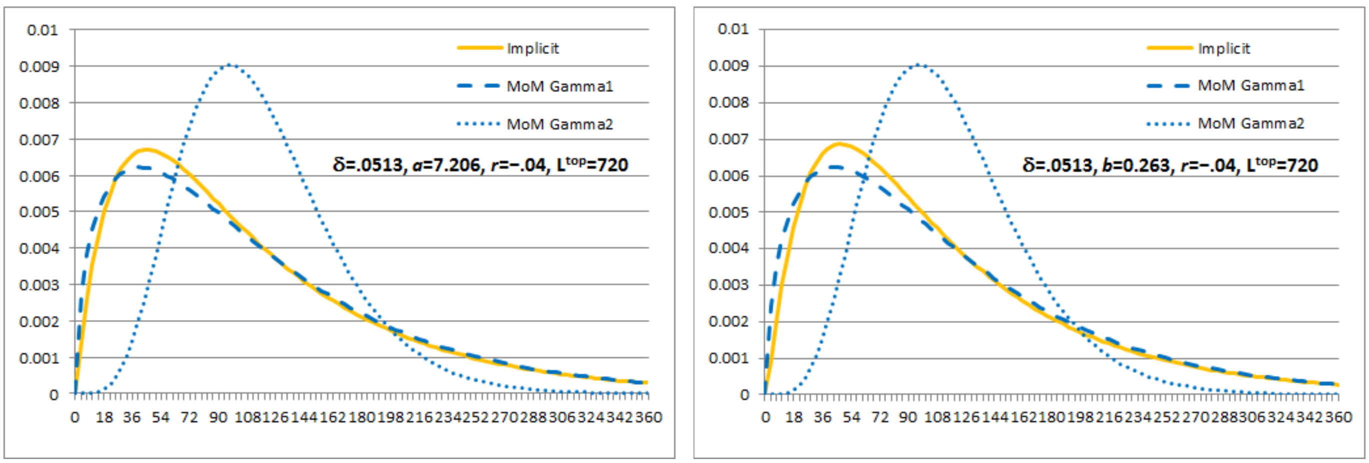
Notes on Figure 12: Decreasing δ to .01 necessitates longer service lives, so the densities' tails are quite thick.

Figures 8-12 all implied an essential compatibility of general individual-level resale-price profiles with cohort-level geometric depreciation (permitting capital-accumulation accounting at the net-stock level, instead of vintage accounting). The necessary connection was always a weighting function that resembles a Gamma density of service-lives. Moreover, the Gamma parameters δ and v (and so, $\bar{L}$) were enough to identify the useful features of cohort-level depreciation. So (2.3) and (2.4) accomplish approximately what (1.6) did exactly, only without disqualifications when individual resale-price profiles are downwardly concave.

Indeed, (2.3) and (2.4) are so flexible that the real burden of upholding a geometric cohort falls on the service-life density. We may reasonably suspect that if some discovered density of service lives is not *close enough* to a Gamma — which accountants must decide for themselves — then the geometric form is probably not adequate to describe the cohort's loss of value. Then, barring Diewert and Wei (2017), separate cohort-by-cohort accounting would be required.

What about *wrong* Gamma densities? So far, I have found two ways to upset the machinery of expression (3.1). The first is to specify a noticeably negative rate of return. In Figure 13, the implicit density that makes (2.3) or (2.4) fit within the stipulated 5-percent geometric cohort has $v \approx 1.57$ (not far from the maintained v of 1.65) but depreciation/rate parameter $\tilde{\delta} \approx .0135$, which resembles the stipulated δ, *plus* r. MoM1, which matches the mean and variance of the implicit density, is close, but MoM2, constrained to the stipulated δ, is shifted well to the right and narrower ($v \approx 6$). Efforts by "scrap-

Figure 13: Negative Rates of Return



Notes on Figure 13: Reducing r to **–**.04 induces implicit and MoM1 service-life distributions that do not reveal the stipulated 5-percent δ. (MoM2 returns the stipulated δ by construction.)

yard economists" to estimate cohort-level depreciation from retirement data would fail in such liquidity-trap conditions. Why? The answer stems from the implicit individual-level depreciation rates of forms (2.3) and (2.4), which vary inversely with r, particularly for negative r, unlike (1.6) or the cohort-level (1.3), where r and $(v–1)/L$ (or r and δ) are strictly independent. But this should not be viewed as a defect of the flexible individual forms. Indeed, for negative r — which effectively includes cases where the inflation rate on new capital goods outstrips the finance rate — we might want the depreciation rate to increase, so as to keep the user-cost (barely) positive.[23]

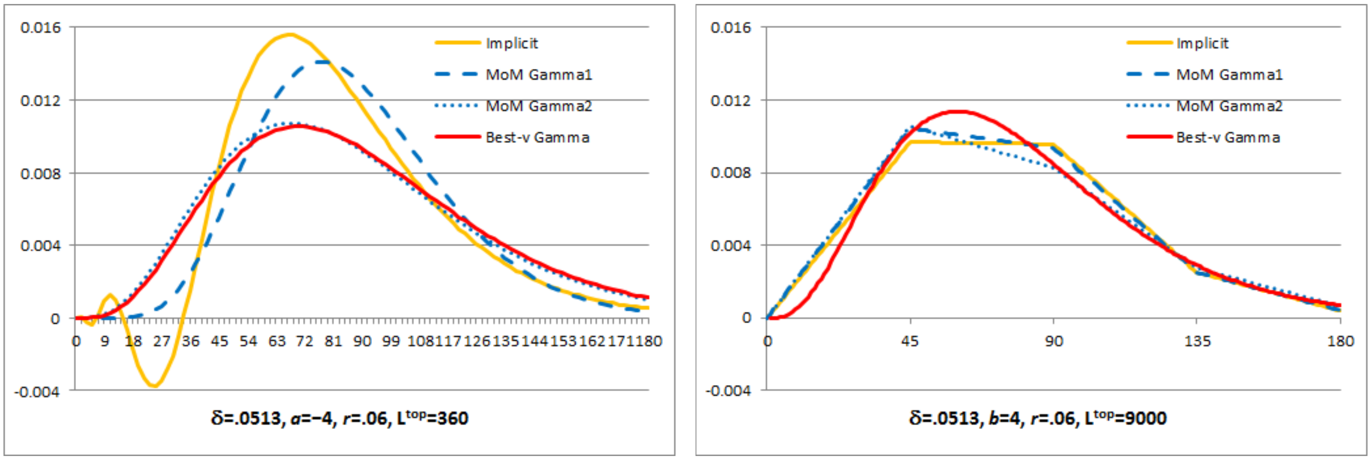
There is also the deeper issue of what really is approximately fixed. I have used (3.1) as a calculation tool, to examine how the implied service-life density changes shape in response to changes in a (or b), r, and δ, each adjusted one at a time. But if actual retirement distributions move only slightly across economic conditions, then it would make sense to adjust parameters in cause-and-effect *pairs*, such that the distribution remains relatively unperturbed. This paper does not conduct those sorts of

[23] A long-ago effort by the BLS to incorporate the implicit rental value of owner-occupied residences into the CPI foundered over this very issue. I thank A. Katz for recounting this old war story. *Cf.* Section 6, below.

experiments — an explicit parameterization of the service-life density in terms of the system's parameters would be useful there — but another one does. (Sliker, 2016)

A secondary problem occurs for sharp decreases in *a* or increases in *b* — retracting individual profiles so far from the one-hoss shay case that they approach "retirement-in-place." In Figure 14, for

Figure 14. Make Individual-Level Resale-Price Profiles Very Convex:



example, both plots show the infeasibility of large numbers of individual resale-price profiles falling faster than the cohort that is supposed to be their weighted average. For *a*-type profiles, negative weights that "snake" result; for *b*-type profiles, the snaking is so violent that only a severe coarsening of the Δ-grid allows (very imprecise) calculation of moments. Fortunately, this is the region where the exact *v*-type decomposition of the geometric cohort works (*cf.* the red curves).

We summarize this long and winding section with a short and direct conclusion: If the cohort depreciates in a geometric fashion, then it is likely that (zero-valued) retirements approximately follow a Gamma distribution. Moreover, except for the troublesome case of negative own rates of return, which raises deeper questions, the parameters of that Gamma distribution seem sufficient to inform the depreciation rate and average service-life of the geometric cohort. The equivalent contrapositive conclusion is: If zero-value retirements do not approximately follow a Gamma distribution, then the cohort probably does not depreciate in a geometric manner. The statements depend on forms (2.3) or (2.4) being "good enough" approximations of individual-level profiles that are generally not observed. The statements do not rule out non-geometric cohorts with Gamma-distributed retirements — e.g. if the values of the individual-level curvature parameter (which I've called *v*) in (1.6) and the Gamma density shape parameter (which I've also called *v*) happen to be different — but they do put a premium on the estimation of retirement patterns.[24]

[24] Professors Diewert and Wykoff (2007) have urged statistical agencies to change "their investment surveys [to] cover… sales and retirements of used assets as well as purchases of new assets."

4. Obsolescence

From the standpoint of capital measurement, the reason for tabulating resale-price and rental-price profiles is to put the value and service-flows, respectively, of old capital on a comparable basis with new, so that Leontief-consistent wealth-stock and productive-stock aggregates may be built. But the meaning of "new" capital is ambiguous. We might imagine restoring an old individual to the same luster, functionality, and years of service it had when it was brand new, but its "as-if-new price" would likely not match the observed prices of *actual* new assets nearer the technological frontier (nor make the restoration effort anything but a labor of love). The ratio of such an as-if-new price to the frontier new price constitutes an unambiguous measure of realized obsolescence.[25] Above, after Figure 7, I termed that ratio $1/B$ — i.e., the (reciprocal of the) degree by which the frontier asset is *better* than the restored one. It would be hard to model a large, unanticipated, one-time obsolescence shock, even after it had plainly occurred, within the framework of smooth changes considered here. One must instead suspend tabulations as of the date of the obsolescing event, multiply the prices and rents of the old vintage by factor $1/B$, then resume standard geometric depreciation, at the customary rate, on the remaining values. Unlike hurricane damage, obsolescence would not remove assets from service until they were completely devalued, because the paper has not allowed for positive scrap values.

In the absence of deep markets for used assets, the measurement of obsolescence might use hedonic comparisons of prices of frontier *versus* next-to-frontier vintages of assets while both are still actively sold as "new" (Copeland, Dunn, and Hall, 2011).[26] Provided values of B stay fixed once set, the succession of vintages allows a multiplicative daisy-chain of obsolescences, such that the as-if-new price of quite old assets may be very low indeed. If the rate of obsolescence were constant and (therefore) expected, then individual rental-price profiles with respect to age would be proportional to:

$$e^{-\ln B(t-m)}\,\phi(s,L) \tag{4.1}$$

where $e^{-\ln B(t-m)}$ enhances $\phi(s,L)$, the individual age-efficiency profile. (The black-font portions of the right sides of equations (2.2), (2.5), and (2.6), above, are three examples before enhancement, though their r values will be modified soon.[27]) The exponential term's arguments represent the constant rate of obsolescence ($\ln B$), the calendar date (t), and the date when the individual's vintage was the new one (m—think "model year"). A key difference between obsolescence (whether or not constant and expected) and "aging" of the usual sort is the influence of one's arbitrary L-value on the rate of aging. By contrast, obsolescence occurs uniformly to all a cohort's surviving members, via its multiplicative effect on their common as-if-new price.

Of course $s = t-m$, so a resale-price profile accounting for constant-rate obsolescence would be proportional to:

[25] Unambiguous because individual resale-price profiles all begin at 1, so dialing differently-lived individuals back to age zero would restore their values to 1, times whatever their common as-if-new price happens to be.
[26] Though second-hand autos constitute one of the two deepest of used asset markets (the other is housing).
[27] And the *blue*-font portions of the right sides of equations (2.2), (2.5), and (2.6), above, are individual user-costs.

$$\int_s^L e^{-r(u-s)}\, e^{-\ln B\, u}\, \phi(u,L)\, \mathrm{du} \tag{4.2}$$

where the constant of proportionality is the reciprocal of the same expression, evaluated at *s*=0. Carrying out the integration[28] and normalization, we can express the enhanced *v*-, *a*-, and *b*- type resale-price profiles, respectively, as:

$$e^{-\ln B(t-m)}\,(1-s/L)^{v-1} \tag{4.3}$$

$$e^{-lnB(t-m)}\,\frac{a\left(1-e^{(r+lnB)(s-L)}\right)-(r+lnB)\left(1-e^{a(s/L-1)}\right)}{a\left(1-e^{-(r+lnB)}\right)-(r+lnB)\left(1-e^{-a}\right)} \tag{4.4}$$

$$e^{-lnB(t-m)}\left(\frac{e^{\frac{(r+\boldsymbol{lnB})s}{1+b}}-e^{\frac{(r+\boldsymbol{lnB})L}{1+b}}}{1-e^{\frac{(r+\boldsymbol{lnB})L}{1+b}}}\right)^{1+b} \tag{4.5}$$

Thanks to the special, constant-rate form of obsolescence here, expressions (4.3)-(4.5) maintain family resemblances with earlier expressions (1.6), (2.3), and (2.4). The effective own rate of return, *r* + *lnB*, exceeds mere *r*, as anticipated obsolescence augments the discounting and drives up the second multiplicative factors in (4.3)-(4.5) — although by less than realized obsolescence drives down the first multiplicative factors. Form (4.3) in particular is still exact, under Gamma weighting, for a Geometric cohort, though the geometric depreciation rate is now δ + *lnB*, while Gamma rate parameter is still only δ, so the excavator of Gamma retirement distributions can still contribute to the measurement of cohort geometric depreciation, but will need some help from a hedonic practitioner.

For completeness, apply the extended user-cost transformation $\left[r-\frac{\partial}{\partial s}-\frac{\partial}{\partial t}\right]$ to each of forms (4.3)-(4.5) to obtain the enhanced rental-price profiles:

$$\left(r+\boldsymbol{ln}B+\frac{v-1}{L}\right)\, e^{-lnB(t-m)}\left(1-\frac{s}{L}\right)^{v-1}\frac{r+\boldsymbol{ln}B+\dfrac{v-1}{L-s}}{r+\boldsymbol{ln}B+\dfrac{v-1}{L}}\quad \ldots v>2 \tag{4.6}$$

$$\frac{(r+\boldsymbol{ln}B)(a-(r+\boldsymbol{ln}B)L)}{a(1-e^{-(r+\boldsymbol{ln}B)L})/(1-e^{-a})-(r+\boldsymbol{ln}B)L}\left(e^{-lnB(t-m)}\,\frac{e^{a\,s/L}-e^{a}}{1-e^{a}}\right) \tag{4.7}$$

$$\frac{r+\boldsymbol{ln}B}{1-e^{\frac{(r+\boldsymbol{lnB})L}{1+b}}}\; e^{-lnB(t-m)}\left(\frac{e^{\frac{(r+\boldsymbol{lnB})s}{1+b}}-e^{\frac{(r+\boldsymbol{lnB})L}{1+b}}}{1-e^{\frac{(r+\boldsymbol{lnB})L}{1+b}}}\right)^{b} \tag{4.8}$$

The blue portions of each expression are the individual user costs. Their expectation over the Gamma distribution of service lives approximates the cohort geometric user cost, *r* + lnB + δ — with the

[28] Because *r* is an argument to *v*- and *b*- type efficiency profiles but not to *a*- type profiles — *Cf.* expressions (2.2) and (2.6) versus (2.5) — it's a good idea to do the exercise for the *a*- type first. Once the result becomes clear, in which *r* + ln*B* replaces *r* alone, one may try the *v*- and *b*- type exercises, but with *r* + ln*B* replacing *r* in the unenhanced efficiency profile *to begin with*. Then deriving the rental and efficiency profiles from the resulting resale-price profile will confirm the *ansatz*.

approximation being exact for (4.6). The black portions of each expression are the enhanced individual age-efficiency profiles, $e^{-\ln B(t-m)}\ \phi(s,L)$ from expression (4.2), with which this exercise began.

5. BEA's Depreciation Rates for Structures

One of this paper's key findings is that the declining-balance rate of a geometrically depreciating cohort of an asset type must exceed 1, for the sake of reasonable depreciation behavior on the part of the cohort's individual members. Yet every structures asset tracked by the Bureau of Economic Analysis is assigned a declining-balance rate that is less than 1. How did this come to pass, and what might be done about it?

Two factors may explain the inconsistency. First, sensible speculation attaches to the land beneath the structure, since the two assets are often sold for a single price. To the extent that land values are not properly purged from the structure+land compound, congestion pricing would disproportionately affect older, close-in structures, both slowing their measured depreciation and even causing apparently increasing rents with age. Second, increasing land values raise the opportunity cost of keeping an old structure, so retirements would happen at still-substantial positive observed prices, speeding the cohort depreciation rate vis-à-vis the rate obtained when assets to be retired are nearly worthless.[29] Either consideration suggests BEA's estimated depreciation of structures ought to be faster.

BEA's structures rates are not new. Dr. Barbara Fraumeni, BEA's former Chief Economist, while investigating one of the Wykoff-Hulten Treasury reports (Wykoff and Hulten, 1979), determined BEA's depreciation rate on "new 1-to-4 unit residential structures" was a simple rounded average of two rates from a Harvard dissertation (Weston, 1972) and two from a research article published by Leigh (1980):

$$d = ¼((.016 + .015) + (.0106 + .0095)) = .012775 \approx .013 \qquad (5.1)$$

Dividing .013 into a DBR of .91, which Fraumeni termed the "default declining balance rate," gives $\bar{L}$=70, confirming BEA's service life previous to the 1996 comprehensive revision of the National Income and Product Accounts (Fraumeni, 2001). When that life was reset to 80 years as part of the comprehensive revision, so was the depreciation rate (to .0114), but the DBR was held fixed. Leigh's estimates are based on geometric perpetual-inventory recursions of housing-start counts between benchmarks derived from the *Census of Housing* for 1950 (revised), 1960, and 1970. (The benchmarks are not the Census counts themselves, but are substantially reduced, in proportion to resale values of homes built in earlier years that survived to the Census years.) Leigh's calibrations do not come with standard errors, but she situates her estimates between bounds available from the literature (i.e., $.0036 \leq \delta \leq .0235$). Yet such bounds are really quite wide, implying (for $\bar{L}$ = 80) v between .288 and 1.88. Using Weston's estimates alone (i.e., .0155) would have pushed v above 1, though he seems not to have adjusted for survival (so I'm a bit surprised that Leigh's results are even slower). I would point out that of the three elements in the geometric-Gamma system — δ, v, and $\bar{L}$ — δ has the greatest exposure to data so far, followed at some

[29] This points up the importance of what this paper has *not* covered: positive floor prices. Preliminary calculations suggest the implied Gamma rate parameter of the *retirement* distribution (for floor prices a flat positive fraction of new prices) would exceed the rate parameter on the (unobserved) distribution of zero-price *lifespans*, with the approximate geometric cohort depreciation rate in between.

distance by $\bar{L}$. Estimates ought not pivot on a constant, and until now murky, *v*.

6. Capital Accounting When Real Rates of Return Are Not Positive

Sections 2 and 3 both pointed out a problem with geometric cohort accounting for $r \le 0$. In ordinary times the problem would barely register academic interest, yet nominal interest rates have been held near zero in America for the past ten years — longer still in Japan. For some asset types, real rates of return are therefore zero or negative. Now, both the *a*- and *b*- type individual-level user-costs stay positive (albeit very near zero) no matter how negative *r* may be. Weighting across individuals by the (near)-Gamma PDFs implicit in the geometric cohort would enable continued, if non-geometric, accounting during real liquidity-trap conditions.[30]

It is particularly noteworthy that the 0-discount limits of the *b*-form individual resale-price and rental-price profiles are:

$$P_b(s,0) = \lim_{r\to 0}\left(\frac{e^{\frac{r\,s}{1+b}}-e^{\frac{r\,L}{1+b}}}{1-e^{\frac{r\,L}{1+b}}}\right)^{1+b} = \left(1-\frac{s}{L}\right)^{1+b} \tag{6.1}$$

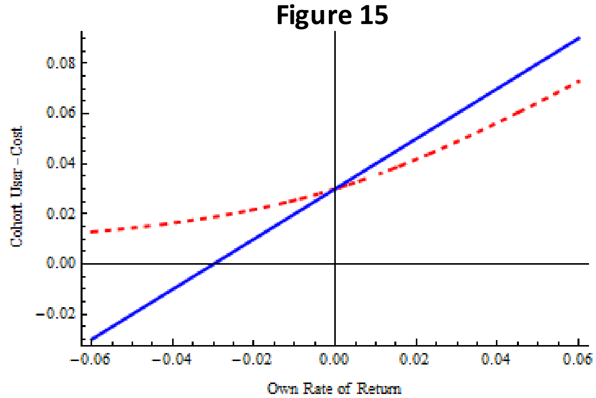
$$R_b(s,0) = \lim_{r\to 0}\ \frac{r}{1-e^{-\frac{r\,L}{1+b}}}\left(\frac{e^{\frac{r\,s}{1+b}}-e^{\frac{r\,L}{1+b}}}{1-e^{\frac{r\,L}{1+b}}}\right)^{b} = \frac{1+b}{L}\left(1-\frac{s}{L}\right)^{b} \tag{6.2}$$

Setting $b = v\text{–}2 > 0$ as $r\rightarrow 0$, then using the Gamma density (1.9), would permit a smooth connection to the realm of negative own rates of return. The implied cohort-level user-cost is then:[31]

$$UC = \int_0^\infty \frac{r}{1-e^{rL/(1-v)}}\,\frac{\delta^{v}L^{v-1}}{\Gamma(v)e^{\delta L}}\,dL\;\; x = \begin{cases} -r\big((1-v)\delta/r\big)^{v}\,\zeta(v,1+(1-v)\delta/r) & r<0 \\ \delta & r=0 \\ r\big((v-1)\delta/r\big)^{v}\,\zeta(v,(v-1)\delta/r) & r>0 \end{cases} \tag{6.3}$$

Comparing the standard geometric (solid blue) and *b*-type (dashed red) cohort user-cost curves (for example values of $v = 2.5$ and $\delta = .03$), we find the alternative form keeps the user cost positive for $r \le 0$:

Figure 15



[30] The trick is to settle on a *fixed* distribution that does not—unlike the calibrations in Section 3 — depend on *r*. The paper's whole theme is that there *is* such a distribution, for which a Gamma, with parameters discoverable by the methods of survival analysis, is a good-enough approximation.

[31] The generalized Riemann Zeta function, $\zeta(v,x) = \sum_{k=0}^{\infty}(k+x)^{-v}$, excluding terms where $k+x=0$. We're interested in the first two choices, as the straightforward cohort geometric form can take care of itself for $r > 0$.

7. Conclusions

This paper has decomposed cohort-level Geometric depreciation into a weighted average of disparate individual patterns of wear-and-tear, overlain by an obsolescence process that operates uniformly across a cohort's members. General forms for individual-level depreciation were presented, and in nearly all cases the weighting function necessary to fit individuals into a Geometric cohort agreed closely with the shape of a Gamma density of individual service-lives. The Gamma density's rate parameter (plus the obsolescence rate) is interpretable as the Geometric rate of depreciation, and the density's shape parameter makes sense of the much-used but little-understood declining-balance rate. Data on individual-asset retirement ages might then inform the parameters of depreciation, even apart from used-price data.

The exception to a straightforward equating of the parameters of Geometric depreciation and Gamma retirements is the case of a negative rate of return. This is already well known as a potential problem for common constructs of the user-cost. Fortunately, one of the individual-level depreciation forms, aggregated by the same Gamma service-life density, offers a viable and numerically-continuous positive-valued extension of the otherwise-Geometric user cost when the own rate-of-return is negative. This is a useful repair for economic measurement in Liquidity-Trap conditions.

Knowing what the declining-balance rate means implies knowing what it forbids. The declining-balance rates used by the Bureau of Economic Analysis for its structures assets are ruled out by this paper's analysis as being too small (<1). Since, in the geometric framework, the declining-balance rate equals the product of the depreciation rate and the average service-life, increasing the former entails increasing one or other of the latter.

References

Copeland, Adam, Wendy Dunn, and George Hall (2011). "Inventories and the Automobile Market." *RAND Journal of Economics*, vol. 42, iss. 1 (Spring): 121-149.

Diewert, Erwin and Hui Wei (2017). "Getting Rental Prices Right for Computers: Reconciling Different Perspectives on Depreciation." *Review of Income and Wealth*, ser. 63, suppl. 1 (February): S149-S168.

Diewert, Erwin and Frank C. Wykoff (2007). "Depreciation, Deterioration and Obsolescence When There is Embodied or Disembodied Technical Change" (revised July 12). Working Paper. Obtainable at https://microeconomics.ca/erwin_diewert/final.pdf.

Fraumeni, Barbara (2001). Personal Communication (January).

Hulten, Charles R. and Frank C. Wykoff (1981). "The Measurement of Economic Depreciation." In *Depreciation, Inflation, and the Taxation of Income from Capital*. Charles R. Hulten, ed. The Urban Institute. Washington, DC: 81-125.

Leigh, Wilhelmina A. (1980). "Economic Depreciation of the Residential Housing Stock of the United States, 1950-1970." *Review of Economics and Statistics*, vol. 62, no. 2 (May): 200-206.

Sliker, Brian K. (2016). "Capital Accounting for Rapidly Obsolescing One-Hoss Shay Individuals in Geometric Cohorts." Paper presented at the 34th General Conference of the International Association for Research in Income and Wealth, Dresden, Germany. August 26.

U.S. Department of Commerce, Bureau of the Census (1966). *Housing Construction Statistics, 1889 to 1964*, Table A-5, pp. 24-25. Digitized at http://archive.org/details/housingconstruct00unit.

U.S. Department of Commerce, Bureau of the Census (1972). *Housing Starts, 1959 to 1971*, Table 2, p. 18. (Available at the Library of Congress: HD9715 .u5 u55 1972).

U.S. Department of Commerce, Bureau of Economic Analysis (2003). *Fixed Assets and Consumer Durable Goods in the United States, 1925-97*. U.S. Government Printing Office.

U.S. Department of the Treasury, Bureau of Internal Revenue (1942). *Bulletin "F": Income Tax Depreciation and Obsolescence, Estimated Useful Lives and Depreciation Rates*. Revised. U.S. Government Printing Office.

Weston, Rafael R. (1972). *The Quality of Housing in the United States, 1929-70* (September). Harvard Economics Department doctoral dissertation.

Winfrey, R. (1935). *Statistical Analysis of Industrial Property Retirements*. Bulletin 125. Iowa State Engineering Experiment Station, vol. XXXIV, no. 28.

Wykoff, Frank C. and Charles R. Hulten (1979). "Tax and Economic Depreciation of Machinery and Equipment: A Theoretical and Empirical Appraisal, Phase II Report." *Economic Depreciation of the U.S. Capital Stock: A First Step*. U.S. Department of the Treasury, Office of Tax Analysis, July 26.

Appendixes

Appendix 1. Exact Gamma forms

To show the main result:

$$e^{-\delta s} = \int_s^\infty (1-s/L)^{v-1}\, \delta^v L^{v-1} e^{-\delta L}/\Gamma(v)\, dL \tag{1.8}$$

keep changing variables until most of the right-hand side can be made to cancel with $\Gamma(v)$. First cancel common powers of L across the individual age-price form and the numerator of the Gamma density, leaving:

$$e^{-\delta s} = \int_s^\infty (L-s)^{v-1}\, \delta^v e^{-\delta L}/\Gamma(v)\, dL.$$

Then replace $\delta(L-s)$ by t, which entails quitting dL for $(dt)/\delta$ and changing the lower limit of integration from $L \to s$ to $t \to 0$. When the dust clears, all the δ's will have canceled, leaving only:

$$e^{-\delta s} = e^{-\delta s} \int_0^\infty t^{v-1} e^{-t}/\Gamma(v)\, dt.$$

But $\int_0^\infty t^{v-1} e^{-t}\, dt$ is how $\Gamma(v)$ is defined, so we're left with $e^{-\delta s}$ on both sides. QED.

To show the mean:

$$v/\delta = \int_0^\infty L\, \delta^v L^{v-1} e^{-\delta L}/\Gamma(v)\, dL \tag{1.10}$$

do much the same. Cancel powers of L, then replace δL by t, and so dL by $(dt)/\delta$, leaving:

$$v/\delta = (1/\delta) \int_0^\infty t^v e^{-t}/\Gamma(v)\, dt.$$

Now $\int_0^\infty t^v e^{-t}\, dt$ evaluates to $\Gamma(v+1)$, and $\Gamma(v+1)/\Gamma(v) = v!/(v-1)! = v$. QED.

Appendix 2. Parametric Individual Depreciation forms

The three parametric individual forms (v-, b-, and a-type) are all distortions of textbook cases. The v- type derives obviously from the straightline individual resale-price function:

$$(1-s/L) \qquad \rightarrow \qquad (1-s/L)^{v-1}$$

and for $v>2$ remedies the straightline form's rental-price shortcomings. (In fact, the case of $v=3$ is the resale-price profile implied by the continuous limit of sum-of-years'-digits depreciation.) But a different distortion:

$$1-(s/L)^\Lambda$$

has no reasonable implied rental-price function for *any* real Λ.

The one-hoss shay individual age-efficiency function:

$$\begin{cases} 1 \; if \; 0 < s < L \\ 0 \; otherwise \end{cases}$$

is better known than its resale-price counterpart:

$$\frac{e^{rs} - e^{rL}}{1 - e^{rL}} = \int_{u=s}^{L} e^{-r(u-s)} \times 1 \, du \Big/ \int_{u=0}^{L} e^{-r(u-0)} \times 1 \, du$$

but it is the counterpart's generalization that establishes the *b*- form:

$$\frac{e^{rs} - e^{rL}}{1 - e^{rL}} \quad \rightarrow \quad \left(\frac{e^{rs/(1+b)} - e^{rL/(1+b)}}{1 - e^{rL/(1+b)}}\right)^{1+b}$$

There is a beauty to the *b*- form that carries over to its implied age-efficiency function:

$$\left(\frac{e^{rs/(1+b)} - e^{rL/(1+b)}}{1 - e^{rL/(1+b)}}\right)^{b} = \frac{\left[r - \frac{\partial}{\partial s}\right]\left(\frac{e^{rs/(1+b)} - e^{rL/(1+b)}}{1 - e^{rL/(1+b)}}\right)^{1+b}}{\left[r - \frac{\partial}{\partial s}\right]\left(\frac{e^{rs/(1+b)} - e^{rL/(1+b)}}{1 - e^{rL/(1+b)}}\right)^{1+b} \quad @s = 0}$$

To reverse course — i.e., to recover the resale-price profile from the age-efficiency form — start with:

$$\int_{s}^{L} e^{-r(u-s)} \left(\frac{e^{rL/(1+b)} - e^{ru/(1+b)}}{e^{rL/(1+b)} - 1}\right)^{b} \mathrm{d}u$$

but change the variable of integration twice. First, replace $e^{ru/(1+b)}$ by "y," so $du = (dy/y)(1+b)/r$, which transforms the integral to:

$$\frac{1+b}{r} \frac{e^{rs}}{(e^{rL/(1+b)} - 1)^{b}} \int_{e^{rs/(1+b)}}^{e^{rL/(1+b)}} \frac{\left(e^{rL/(1+b)}/y - 1\right)^{b}}{y^{2}} \mathrm{d}y$$

Then replace $e^{ru/(1+b)}/y - 1$ by "z," so $dz = -e^{ru/(1+b)} dy/y^2$, which transforms the integral to:

$$\frac{1+b}{re^{rL/(1+b)}} \frac{e^{rs}}{(e^{rL/(1+b)} - 1)^{b}} \int_{0}^{\frac{e^{rL/(1+b)}}{e^{rs/(1+b)}} - 1} \mathrm{z}^{b} \mathrm{d}z$$

This plays out as:

$$\frac{(e^{rL/(1+b)} - e^{rs/(1+b)})^{1+b}}{re^{rL/(1+b)} (e^{rL/(1+b)} - 1)^{\mathrm{b}}}$$

Divide by the same expression evaluated at s=0 to get (2.4).

For the *a*- type individual efficiency profile, first "trap" the geometric form, forcing it to zero at age s=L, then normalize the result to equal 1 at age 0, and finally change parameter δ to $-a/L$:

$$e^{-\delta s} \quad \rightarrow \quad e^{-\delta s} - e^{-\delta L} \quad \rightarrow \quad (e^{-\delta s} - e^{-\delta L})/(1 - e^{-\delta L}) \quad \rightarrow \quad (e^{as/L} - e^{a})/(1 - e^{a})$$

The last step makes the efficiency profile s/L-homogeneous, so individuals with different lifespans might have the same basic shape, but it implies long-lived individuals deteriorate and depreciate more slowly than short-lived ones. It also distinguishes the efficiency parameter a from the own-interest rate r in the resale-price form, which is *not* s/L-homogeneous. To find that form, integrate:

$$\int_{s}^{L} e^{-r(u-s)} \frac{e^{au/L} - e^{a}}{1 - e^{a}} \mathrm{du}$$

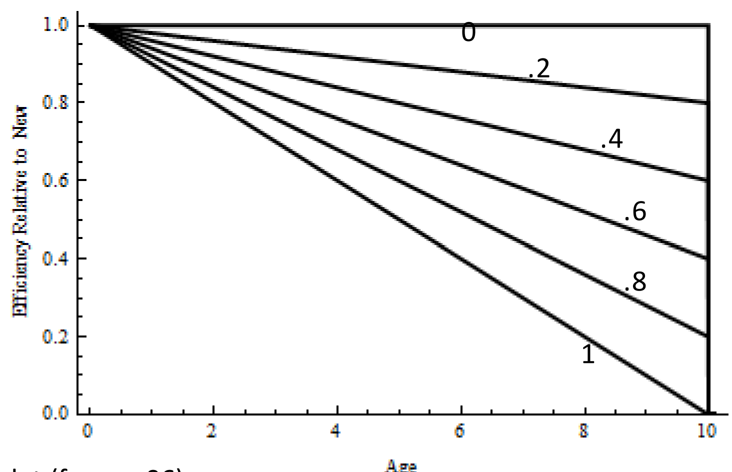
which is straightforward. Divide by the same result evaluated at s=0 to get (2.3).

Appendix 3. Linear Efficiency Declines and Near-Gamma Densities

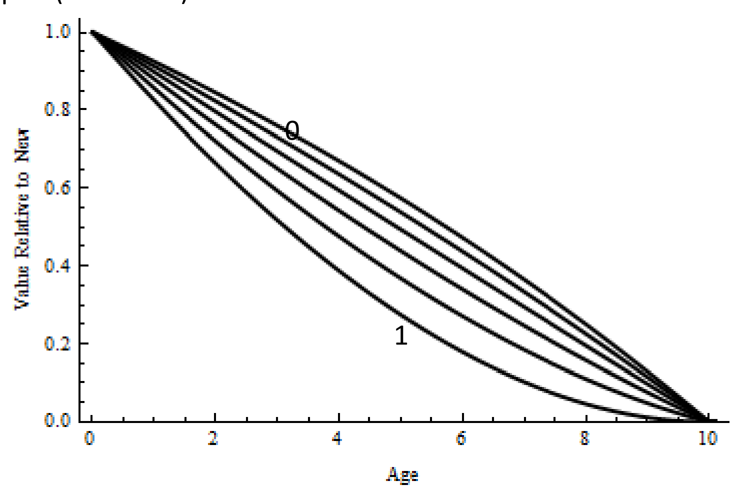
Suppose an individual asset's age-*efficiency* profile declines linearly, "hits the wall" at its lifespan, and then drops to 0:

$$\phi_i = 1 - g\, s/L_i$$

for values of the common slope parameter g between the one-hoss shay (g=0) and straightline efficiency-loss (g=1) limits:



Resale-price profiles plot (for r = .06) as:



from the derived resale-price profile:

$$\frac{(g-rL)(e^{rs}-e^{rL})+grL\left(e^{rs}-\frac{s}{L}e^{rL}\right)}{(g-rL)(1-e^{rL})+grL}$$

Now to find the PDF of L that reconciles the individual price profiles with the cohort $e^{-\delta s}$:

$$e^{-\delta s}=\int_s^\infty \frac{(g-rL)(e^{rs}-e^{rL})+grL\left(e^{rs}-\frac{s}{L}e^{rL}\right)}{(g-rL)(1-e^{rL})+grL}\boldsymbol{f}(\boldsymbol{L})dL$$

differentiate with respect to *s* twice:

$$\delta^2 e^{-\delta s}=r^2e^{rs}\int_s^\infty \frac{g-rL+grL}{(g-rL)(1-e^{rL})+grL}f(L)dL+\frac{(1-g)r^2s\,e^{rs}}{(g-rs)(1-e^{rs})+grs}f(s)$$

and a third time:

$$\begin{aligned}-\delta^3e^{-\delta s}=r^3e^{rs}&\int_s^\infty \frac{g-rL+grL}{(g-rL)(1-e^{rL})+grL}f(L)dL\\&+\left\{\frac{(1-g)(1+rs)-g}{(g-rs)(1-e^{rs})+grs}-\frac{(1-g)rs-(1-e^{rs})}{[(g-rs)(1-e^{rs})+grs]^2}(1-g)rs\right\}r^2e^{rs}f(s)\\&+\frac{(1-g)s}{(g-rs)(1-e^{rs})+grs}r^2e^{rs}f'(s)\end{aligned}$$

Substitute from the second derivative into the third to replace the integral, leaving an ordinary differential equation:

$$\begin{aligned}(r+\delta)\delta^2e^{-\delta s}=&\frac{-(1-g)r^2s\,e^{rs}}{(g-rs)(1-e^{rs})+grs}f'(s)\\&-\left\{\frac{1-2g}{(g-rs)(1-e^{rs})+grs}+\frac{1-e^{rs}-(1-g)rs}{[(g-rs)(1-e^{rs})+grs]^2}(1-g)rs\right\}r^2e^{rs}f(s)\end{aligned}$$

This has the general solution:

$$f(L)=\left[\left(r-\frac{g}{L}\right)(1-e^{-rL})+gre^{-rL}\right]L^{\frac{g}{1-g}}\left\{\boldsymbol{c}+\frac{(r+\delta)\delta^{\frac{1}{1-g}}}{(1-g)r^2}\,\Gamma\left(1-\frac{g}{1-g},\delta L\right)\right\}$$

...where ***c*** is a constant of integration. That constant needs to be 0 for $\int_0^\infty f(L)dL=1$. For *g*=0, *f*(*L*) reduces to $(\delta/r)(r+\delta)(1-e^{-rL})e^{-\delta L}$; and for *g*=1, we obtain $(\delta^2/r^2)(r+\delta)(e^{-rL}-1+rL)e^{-\delta L}$. For strictly interior values of *g*, the density cannot shake the Incomplete Gamma function, which is accessible indirectly in *Excel* by subverting the cumulative form of that package's GAMMA.DIST function. The algebraic form of *f*(L) is obviously more complicated than a Gamma density, yet plots of the two for the same mean and variance are very close.